Report on the outcomes of a Short-Term Scientific Mission[1]

Action number: CA18232

Grantee name: Dragoș MANEA

Details of the STSM

Title: Control, numerics and asymptotics for parabolic equations on metric graphs – 2 months in Erlangen, Germany

Start and end date: 01/05/2023 to 29/06/2023

Description of the work carried out during the STSM

Description of the activities carried out during the STSM. Any deviations from the initial working plan shall also be described in this section.

During the period of the STSM, I worked together with Nicola De Nitti (PhD student, supervised by Prof. Enrique Zuazua), aiming to develop and control a numerical scheme for the convection-diffusion equation on a metric tree, with boundary controls. The work was split into multiple directions of study:

Direction 1: We studied the relevant literature on numerical schemes for equations on networks, focusing in particular on the adjustments that are required for accommodating Kirchhoff coupling conditions at joints. We developed a fully discrete finite differences scheme for convection-diffusion equations on metric trees, we proved its convergence and computed the order of convergence. One of our findings was the fact that, essentially due to the presence of junctions, we obtain a numerical viscosity term, which places our problem in the setting of discrete parabolic equations, and not in that of hyperbolic equations.

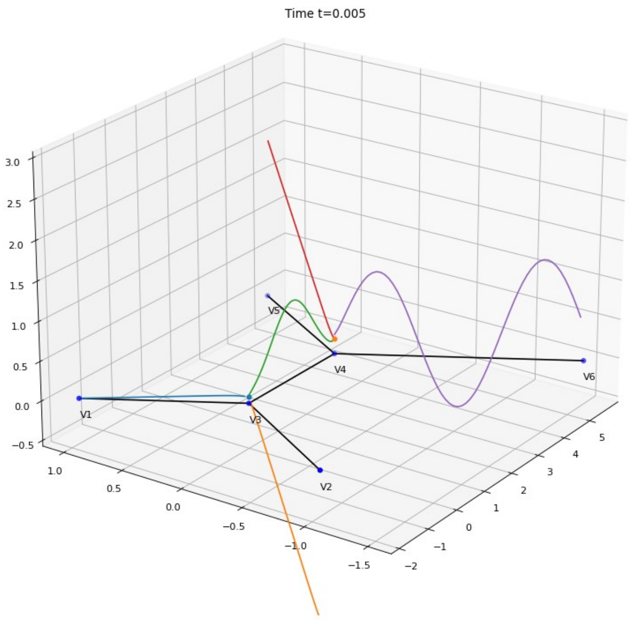
Direction 2: We studied the literature regarding the control of discrete parabolic equations and realised that even the discrete convection equation on a segment was not covered. We first showed that the equation is exactly controllable in any time by constructing an explicit boundary control in this case. However, this approach presented at least three issues: (1) the calculated control might not be optimal; (2) this method cannot be adapted to the presence of a viscosity term; (3) this approach cannot be extended to networks.

Direction 3: With the help of Nicola De Nitti, I familiarised myself with the topic of control theory for linear PDEs and with Carleman inequalities, as presented in the book [1] and the paper [2]. Since the

[1] This report is submitted by the grantee to the Action MC for approval and for claiming payment of the awarded grant. The Grant Awarding Coordinator coordinates the evaluation of this report on behalf of the Action MC and instructs the GH for payment of the Grant.

approach outlined above in “Direction 2” had limited outcomes, we pursued our study of Carleman inequalities in discrete settings, starting with the analysis done in [3]. We got familiarised with discrete calculus (discrete differentiation and integration by parts formulas) and we also extended the integration by parts rule to discrete operators on networks. This is the framework in which we aim to prove our new Carleman estimates: first in the case of a segment, and then on graphs. Finally, we aim to deduce null controllability results from the Carleman inequalities, together with estimations regarding the cost of controllability.

Direction 4: We implemented the fully discrete finite difference scheme in Python and created an animated plot for the solution of the convection-diffusion equation. The visual behaviour of the plot empirically supports the theoretical result about the convergence to the continuous problem. The implementation can be found in the following GitHub repository: Numerical graphs and one example of output is in the picture below:



Description of the STSM main achievements and planned follow-up activities

Description and assessment of whether the STSM achieved its planned goals and expected outcomes, including specific contribution to Action objective and deliverables, or publications resulting from the STSM. Agreed plans for future follow-up collaborations shall also be described in this section.

First and foremost, the STSM was a great opportunity for me to integrate for two months into a dynamic research environment, where I got to know many young researchers and students with whom I established fruitful connections. The internship gave me the occasion to see how people at the Chair for Dynamics, Control, Machine Learning and Numerics work in their daily research activity and observe their interaction and teamwork as an insider.

Mathematically speaking, the main quantifiable achievement of the STSM is the development of the finite difference scheme, its implementation and the proof of its rate of convergence to the continuous problem. In addition, we correctly fitted our problem in the setting of discrete parabolic equations.

I also developed a good understanding of the various methods used in the control theory of discrete systems and studied their transferability in the setting of networks. This knowledge I accumulated will *be useful for future theoretical and numerical studies concerning various types of equations that are posed on networks, leading also to practical applications.*

We also developed a working plan for the continuation of my work with Nicola De Nitti, which will be carried out, remotely, over the following months, in order to publish one or more academic papers on the subject of the project.

This STSM was also a chance for me to enlarge my academic network and start at least two other collaborations with researchers at the FAU – namely Prof. Enrique Zuazua and Dr. Timothée Crin-Barat – on PDE topics that are close to the objectives of the Action. In addition, the discussions with people at Erlangen helped me broaden my view on various mathematical subjects, especially numerical analysis and also machine learning.

References:

[1] J.-M. Coron. *Control and nonlinearity*, volume 136 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2007.

[2] J.-M. Coron and S. Guerrero. *Singular optimal control: a linear 1-D parabolic-hyperbolic example*. Asymptot. Anal. 44 (2005), no. 3-4, 237–257.

[3] F. Boyer, F. Hubert, and J. Le Rousseau. *Discrete Carleman estimates for elliptic operators in arbitrary dimension and applications*. SIAM J. Control Optim., 48(8):5357–5397, 2010.