Report on the outcomes of a Short-Term Scientific Mission[1]

Action number: CA18232

Grantee name: Catherine Drysdale

Details of the STSM

Title: Quasi-basis in Fluid Mechanics

Start and end date: 12/07/2023 to 30/07/2023

Description of the work carried out during the STSM

Description of the activities carried out during the STSM. Any deviations from the initial working plan shall also be described in this section.

The whole purpose of the STSM was to prove existence and uniqueness of an equation in a space that allows for a quasi-basis. The quasi-basis can be described as follows;

- *Definition (Quasi-Basis, as defined). Given a suitable dense subspace X_1 of a Hilbert space H. Let $E_L = \{e_n\}_{n=0}^{\infty}$ and $E_{L^\dagger} = \{e_n^\dagger\}_{n=0}^{\infty}$ be two orthogonal sets such that all $e_n \in X_1$ and all $e_n^\dagger \in X_1$. E_L and $E_{L^\dagger}$ are quasi-bases if, for all $f, g \in X_1$,*

$$\langle f, g\rangle = \sum_{n=0} \langle f, e_n\rangle \langle e_n^\dagger, g\rangle = \sum_{n=0} \langle f, e_n^\dagger\rangle \langle e_n, g\rangle$$

We suspect that quasi-bases can be useful in numerical applications, particularly with non-self-adjoint operators, which often encounter numerical issues such as spectral pollution owning to cut-off errors. In short, if the eigenvectors of a linear operator form quasi-bases for a dense subspace, then differentiation matrices will be able to be constructed using the fact that you can expand the functions meaningfully (i.e., the projections onto the eigenvectors are finite) in the eigenvectors of the non-self-adjoint operators.

By proving the existence of uniqueness of an equation in a space that allows for a quasi-bases, we hoped to provide a theoretical backbone to doing numerics that profit from the quasi-basis such as the computation of spectra and the expansion of solutions in this basis.

In the following section, I describe the test-case in more detail as well as the achievements which resulted in proofs regarding the test case. I sought Professor Rhandi's functional analysis expertise as it was outside of my own area. There were no deviations from the original working plan apart from some delay in the start date of the STSM.

[1] This report is submitted by the grantee to the Action MC for approval and for claiming payment of the awarded grant. The Grant Awarding Coordinator coordinates the evaluation of this report on behalf of the Action MC and instructs the GH for payment of the Grant.

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Description of the STSM main achievements and planned follow-up activities

Description and assessment of whether the STSM achieved its planned goals and expected outcomes, including specific contribution to Action objective and deliverables, or publications resulting from the STSM. Agreed plans for future follow-up collaborations shall also be described in this section.

(max. 500 words)

We determined semigroup properties of the following operator $L: D(L) \to L^2(\mathbb{R})$

$$L = \frac{\partial^2}{\partial x^2} + U\frac{\partial}{\partial x} + \left(\frac{U^2}{4} + \sqrt{c_2} - c_2 x^2\right)$$

where the domain of the operator is given by $D(L) := \{ u \in H^2(\mathbb{R}) : \|(1 + c_2 x^2)\, u\|_2 < \infty\}$ *in the following space*

$$D(L_|) = D(L) \cap Y$$

where $Y = \{ u \in L^2(\mathbb{R}) : \left\|e^{-\frac{U}{2}x}u\right\| + \left\|e^{\frac{U}{2}x}u\right\| < \infty$, *i.e. we were proving that the part of* L *in* Y *generated a strongly-continuous semigroup. We let* Y *be a Hilbert Space with norm defined by the following inner product*

$$\|\cdot\|_Y = \sqrt{\langle \left(e^{-\frac{U}{2}x} + e^{\frac{U}{2}x}\right) \ \cdot\ , \left(e^{-\frac{U}{2}x} + e^{\frac{U}{2}x}\right) \ \cdot\ , \rangle}.$$

We firstly proved that Y *was invariant under* e^{tL} *and following this we proved the continuity property* $\lim_{t\to 0}\|(e^{tL}u - u)\|_Y$ *. This was possible owing to an explicit representation of the integral kernel.*

Following this, we also proved existence and uniqueness of the equation,

$$\frac{du}{dt} = Lu - u^3 + \delta u$$

in the space $X = H^1 \cap Y$.

The proofs above for me provided a meaningful appendix to the bases “Quasi-bases in Fluid Dynamics for Numerical Application”, and I could not have done some of the more technical parts of the proof without Professor Rhandi’s guidance and knowledge of functional analysis. This paper is currently being written, with the responsibility of the first draft and computations falling on me exclusively and then with feedback by Professor Rhandi. We wish the publish this paper on arxiv and following this the International Journal of Applied Mathematics. The computations have thus far been promising, and consequences of the quasi-basis can be seen in a choice of normalisation for higher order terms when doing weakly nonlinear expansions for non-self-adjoint operators. The derivations of the differentiation matrices have been done also. Numerical analysis support is being sought from Lyonell Boulton who will be able to give context particularly on the computational issues of non-self-adjoint systems, which will form a meaningful part of the introduction.

The outcomes were as expected, and in line also with the philosophy of COST in terms of bringing researchers together. An unexpected outcome was the realisation that another area of Professor Rhandi’s research, notably the positive operator semigroups for delay differential equations would provide me with a nice numerical decomposition computing the pseudospectra of delay differential equations. I plan to use my own research funds to explore this with Professor Rhandi later in the year.