**Report on the outcomes of a Short-Term Scientific Mission**

**Action number: CA18232**

**Grantee name: Stojanow, Johannes**

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| **Details of the STSM**  Title: Time-Optimal Control of Parabolic Differential Equations in Non-Reflexive Banach Spaces  Start and end date: 16/08/2023 to 31/08/2023 |
| **Description of the work carried out during the STSM**  Description of the activities carried out during the STSM. Any deviations from the initial working plan shall also be described in this section. |
| *(max. 500 words)*  The first two days were devoted to initial discussions and preparation of a working plan for the subsequent two weeks as well as reviewing existing literature. Due to failures of analytical techniques to establish the Bang-Bang property for the time-dependent case in reflexive Banach spaces, a decision was made to shift the focus on this non-autonomous case before moving on to autonomous problems in non-reflexive spaces. The first week was then used to author and revise a draft on the time-optimal control problem paper and corresponding Bang-Bang phenomenon for time-independent control systems with distributed control on reflexive Banach spaces. The underlying notion of controllability has been better adapted to the problem and the literature. The results were put into context with pre-existing literature on observability and controllability in Banach spaces. An application in Banach spaces except those of integrable, square-integrable or essentially bounded functions was developed in order to demonstrate applicability of the results where previous works failed and linking them to results on controllability of the heat equation in Banach spaces of functions on unbounded domains. An example for bounded domains is in progress. Due to works on the equivalence between time- and norm-optimal controllability, e.g. "On the Equivalence of Minimal Time and Minimal Norm Controls for Internally Controlled Heat Equations" (Wang and Zuazua, 2012), the treatment of norm-optimal control problems had been uncared-for. Frequent discussions on relevant examples and problem classes to extend the results led to ideas for parabolic equations governed by non-autonomous elliptic operator families acting on spaces of functions with unbounded domain as well as treating heat equations with unbounded potential.  It has been discovered, that pre-existing arguments for the autonomous case do not carry over to the non-autonomous case as easily as expected. The wide applicability of theory on non-autonomous systems also made it more imperative to establish the Bang-Bang-Property for this kind of problems first before going for the non-reflexive case.  During the second half of the stay, a notion of the weak backward uniqueness property has been studied. This property is intimately connected with the Bang-bang property, and it is satisfied in some examples of non-autonomous control problems, which gave us a path of proving the Bang-Bang proporty for a (limited) class of non-autonomous problems. |
| **Description of the STSM main achievements and planned follow-up activities**  Description and assessment of whether the STSM achieved its planned goals and expected outcomes, including specific contribution to Action objective and deliverables, or publications resulting from the STSM. Agreed plans for future follow-up collaborations shall also be described in this section.  *(max. 500 words)*  Due to the shift of focus from the non-reflexive case to the non-autonomous case in reflexive Banach spaces, results on the former problem remain undiscovered at this moment. The generalization of Bang-Bang controllability to Banach spaces has been successful and the examples shows that the results fit nicely into those on controllability of parabolic systems in reflexive Banach spaces showing that ellipticity and thickness of the control set in the domain not only lead to null-controllability but also directly to the Bang-Bang-Property of time-optimal controls. It is expected but no yet worked out in detail, that this also holds for elliptic and non-autonomous control families. Using the notion of (cost-uniform approximate) null-controllability on measurable sets in time motivated by "Observability for Non-Autonomous Systems" (Bombach, Gabel, Seifert and Tautenhahn, 2023) and the weak backward uniqueness property mentioned above, the existence of optimal control is expected to hold which poses the critical step to obtain Bang-Bang controls in the non-autonomous case.  In order to cover more fields of application, a future follow-up collaboration has been planned regarding heat and Schrödinger equations with bounded and unbounded potentials building upon the recent work "Spectral Inequality with Sensor Sets of Decaying Density for Schrödinger Operators with Power Growth Potentials" (Dicke, Seelmann and Veselic, 2022). |