A SURVEY ON ONE-PARAMETER OPERATOR SEMIGROUPS BEYOND STRONG CONTINUITY

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Abstract. The theory of C_0-semigroups is a classical tool to study linear autonomous evolution equations in Banach spaces. Yet, there are many evolution equations whose solution semigroup does not satisfy the C_0-property. To study them, a large variety of semigroup theories has been developed beyond the C_0-theory. In this article, we survey those developments and discuss and compare different types of semigroup theories.

Contents

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1. Introduction

One-parameter semigroups of operators provide a framework to study the solutions to linear autonomous evolution equations. Let A be an (in general, unbounded) operator on a Banach space X. We are interested in the solution $u : [0,\infty) \to X$ of the evolution equation

$$\dot{u}(t) = Au(t) \quad \text{for } t \in [0,\infty),$$
$$u(0) = u_0,$$

where $u_0 \in X$. If the solutions (in a suitable sense) exist for all $u_0 \in X$, then – due to the linearity of the problem and the fact that A does not depend on the time variable – one expects that for each $t \in [0,\infty)$ the map $T(t) : X \to X$ that maps u_0 to the values $u(t)$ of the solution at time t, is linear and satisfies the properties $T(s+t) = T(s)T(t)$ and $T(0) = \mathrm{id}_X$. Moreover, if the solution at time t depends continuously on the initial value, then $T(t)$ will be a bounded operator on X.

The family $(T(t))_{t\geq 0}$ is a *one-parameter operator semigroup* or, as we shall briefly say, a *semigroup*. A major point of interest is the regularity of the solutions u with respect to the time variable t. In general, one cannot expect $u(t)$ to be differentiable with respect to t for every initial value (and hence, concepts such as *mild solutions* rather than *classical solutions* enter the stage). However, for a large class of concrete evolution problems, the solutions u are continuous mappings from $[0,\infty)$ to X. In other words, the semigroup $(T(t))_{t\geq 0}$ is a strongly continuous mapping from $[0,\infty)$ into the space $\mathcal{L}(X)$ of bounded linear operators on X. Such strongly continuous semigroups – or, for short, C_0*-semigroups* – have been studied in great detail since the middle of the 20th century. We refer, for instance, to the monographs [48, 65, 72, 109] and the introductory textbook [49] for a thorough treatment of the theory.

However, there is also a large class of evolution equations for which the solution does not depend continuously (with respect to the norm on X) on the time variable t. There are various theoretical frameworks to study such semigroups and hence go beyond the theory of C_0-semigroups. This survey gives an overview of several of them.

Organization of the survey. The structure of the paper is as follows. In Section 2, we motivate by several concrete examples why one would like to develop the theory of operator semigroups beyond the limits of C_0-semigroups. In Section 3, we begin with a gentle introduction, revisiting linear autonomous evolution equations in finite dimensions. We aim to stress the interrelation of three central objects – the semigroup, its generator, and its resolvents – which will be very helpful to guide us through the infinite dimensional realms later. In Section 4, we set the stage for the infinite-dimensional theatre by recalling some essentials of the C_0-semigroup theory. Readers well-versed in C_0-semigroups can safely skip Sections 3 and 4. Yet, we chose to include them for those readers who might need other types of semigroups in their research without first studying the field of C_0-semigroups in detail. Section 5 delves into the core of the article, providing an overview of various semigroup concepts beyond C_0-semigroups. In Section 6, we focus on a specific class of semigroups: those that leave the positive cone in a function space or Banach lattice invariant. In the C_0-case, positive semigroups entail an intriguing theory, and we elaborate on how certain features from this theory remain true beyond the C_0-case.

2. Equations with low time regularity: motivating examples

2.1. **Parabolic equations.**
If one considers parabolic equations of the form

$$\dot{u}(t) = Au(t),$$
$$u(0) = u_0$$

on the spatial domain $\mathbb{R}^d$, where A is a second-order elliptic differential operator, semigroups that are not strongly continuous naturally occur in a number of contexts.

Even for the heat equation – i.e., if A is the Laplace operator Δ – the solution only depend continuously on the initial value u_0 if one works in a sufficiently "small" function space:

Example 2.1 (The heat semigroup on $C_b(\mathbb{R}^d)$)**.** Let $C_b(\mathbb{R}^d)$ denote the space of bounded continuous scalar-valued functions on $\mathbb{R}^d$. For a given initial function $u_0 \in \mathbb{R}^d$, the solution $u : [0,\infty) \to C_b(\mathbb{R}^d)$ of the heat equation

$$\dot{u}(t) = \Delta u(t),$$
$$u(0) = u_0$$

is given by $u(t) = k_t \star u_0$, where $k_t : \mathbb{R}^d \to \mathbb{R}$ is the *heat kernel* at time t. For general $u_0 \in C_b(\mathbb{R}^d)$ ones does not have $u(t) \to u(0)$ with respect to the sup norm as $t \downarrow 0$.

For the heat equation $\dot{u}(t) = \Delta u(t)$ this issue can be solved by focusing on different function spaces such as, for instance, $L^p(\mathbb{R}^d)$ for $p \in [1,\infty)$. But for more general elliptic operators – in particular those with unbounded coefficients, see for instance [102] – it is quite natural to have solutions that do not converge uniformly to the initial value as $t \downarrow 0$.

2.2. **Koopman semigroups on large state spaces.**
Koopman semigroups are semigroups that describe the behaviour of a dynamical system on a state space X by an associated action on a function space. Let us describe the precise notion in the following two definitions:

Definition 2.2 (Semiflows)**.** Let Ω be a topological space. A family $\big(\varphi(t)\big)_{t\in[0,\infty)}$ of continuous mappings $\varphi(t) : \Omega \to \Omega$ is called a *semiflow* on Ω if $\varphi(0) = \mathrm{id}_\Omega$ and $\varphi(s+t) = \varphi(s) \circ \varphi(t)$ for all $s,t \in [0,\infty)$.

Definition 2.3 (Koopman operators and semigroups)**.** Let Ω be a topological space and let $C_b(\Omega)$ denote the space of all bounded and continuous scalar-valued functions in Ω, endowed with the sup norm.

(a) For a continuous map $\varphi : \Omega \to \Omega$ we call the bounded linear operator $T_\varphi : C_b(\Omega) \to C_b(\Omega)$ given by

$$f \mapsto f \circ \varphi$$

the *Koopman operator* or *composition operator* associated to φ.

(b) Let $\big(\varphi(t)\big)_{t\in[0,\infty)}$ be a semiflow on Ω. The family of Koopman operators $\big(T_{\varphi(t)}\big)_{t\in[0,\infty)}$ is called the *Koopman semigroup* or the *composition semigroup* associated to the semiflow.

The Koopman semigroup of a semiflow satisfies $T_{\varphi(0)} = \mathrm{id}_{C_b(\Omega)}$ and $T_{\varphi(t+s)} = T_{\varphi(t)} T_{\varphi(s)}$ for all $s,t \in [0,\infty)$ – i.e., the Koopman semigroup is itself a semiflow on the space $C_b(\Omega)$.

Note that we did not assume that a semiflow satisfies any time regularity property. But even if the mapping $[0,\infty)\times\Omega \to \Omega$, $(t,x) \mapsto \varphi(t)(x)$ is jointly continuous, this does not imply good time continuity properties of the Koopman semigroup. As a simple example, one can demonstrate this by the shift on $\mathbb{R}$:

Example 2.4 (The shift semigroup on the real line)**.** Define a semiflow $\big(\varphi(t)\big)_{t\in[0,\infty)}$ on $\mathbb{R}$ by setting $\varphi(t)(x) := x + t$ for all $x \in \mathbb{R}$ and $t \in [0,\infty)$. Then the associated Koopman semigroup on $\mathrm{C_b}(\mathbb{R})$ shifts all functions to the left.

Clearly, the function $[0,\infty) \times \mathbb{R} \to \mathbb{R}$, $(t,x) \mapsto \varphi(t)(x) = t + x$ is jointly continuous. However, if $f \in \mathrm{C_b}(\mathbb{R})$ is not uniformly continuous, one can easily check that the mapping

$$\begin{aligned} [0,\infty) &\to \mathrm{C_b}(\mathbb{R}), \\ t &\mapsto T_{\varphi(t)} f = f(\,\cdot\, + t) \end{aligned}$$

is not continuous with respect to the sup norm on $\mathrm{C_b}(\mathbb{R})$.

The problem with the shift semigroup could be resolved by considering smaller function spaces, for instance, the space of uniformly continuous and bounded functions. However, this no longer works if some of the mappings $\varphi(t)$ are not uniformly continuous themselves. This makes a theory of Koopman semigroups that does not rely on strong continuity desirable. A step towards such a theory is presented in [57].

2.3. **Transport processes on graphs.** Dynamic processes that are described by shifts as in Example 2.4 can be generalized to network structures.

Example 2.5 (Network flows)**.** Consider a finite, directed graph (V, E), where V denotes the set of all vertices and E denotes the set of all edges. We assign a length $\ell_e \in (0,\infty)$ to each edge $e \in E$. Then we identify each edge e with the interval $[0, \ell_e]$ where 0 corresponds to, say, the initial point of the edge and ℓ_e to the terminal point of the edge (but those choices are, of course, arbitrary, and are sometimes written down differently in the literature). Consider, at each time $t \geq 0$, a function $u_e(t) : [0, \ell_e] \to \mathbb{R}$ on the edge e which is subject to a transport process with a fixed velocity $v_e \in (0,\infty)$. This process is described by the partial differential equation

$$\dot{u}_e(t) = -v_e u_e'(t),$$

where $u_e'(t)$ denotes the spatial derivative and $\dot{u}_e(t)$ denotes the derivative with respect to t (which is computed in a suitable Banach space).

To take the graph structure into account one assumes that, at each vertex v, all the mass that enters v is immediately redistributed to the outgoing edges. If v has more than one outgoing edge, one needs to assign weights to decide which one the mass gets distributed to. This redistribution of mass can be encoded by boundary conditions that one adds to the differential equations for the u_e. This way one ends up with an evolution equation on a state space that is the sum of functions spaces over the intervals $[0, u_e]$. For instance, if we are interested in the behaviour with respect to the L^1-norm, we consider the space $\oplus_{e\in E} L^1([0, \ell_e])$.

A concrete example of such a graph is shown in Figure 1. In this example, only the vertices 2 and 5 have more than one outgoing edge and hence, we only need to assign weights to these outgoing edges. For all vertices with only one outgoing edge, the weight is tacitly assumed to be 1.

Such transport processes on graphs are sometimes referred to as *network flows*. A detailed study on them by means of C_0-semigroups goes back to the seminal paper [83]. Afterwards, the topic was studied in a great number of papers. To name just a few examples, we mention space-dependent velocities [101], infinite networks [42, 43] and delays in the vertices [23].

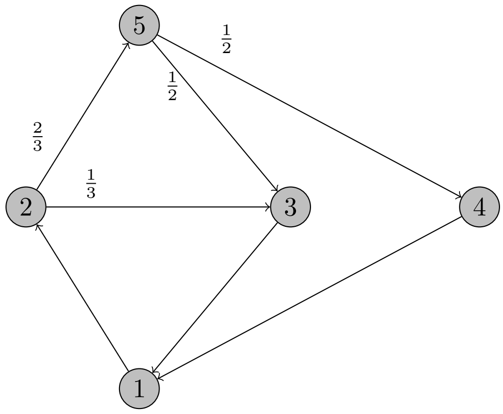


FIGURE 1. A transport process on a directed graph

If one studies the flow on state spaces over the edges that are less well-behaved – for instance on L^∞ [33] or on spaces of measures [39] – one loses the strong continuity. Hence, tools beyond C_0-semigroup theory become necessary to study network flows on such transport processes.

2.4. **Abstract constructions that destroy continuity.** While there is a well-developed theory of strongly continuous semigroups (see Section 4 for details), this class of semigroup is not stable under a number of standard constructions. We discuss a few such constructions in the following examples. Throughout those examples, we freely use notions from the theory of C_0-semigroups that are explained in Section 4.

Example 2.6 (Dual semigroups)**.** Let $T = (T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. The family of dual operators $T' = (T(t)')_{t\in[0,\infty)}$ is also a C_0-semigroup if X is reflexive, but it need not be a C_0-semigroup if X is non-reflexive. For instance, the left shift semigroup on $L^1(\mathbb{R})$ is a C_0-semigroup, but its dual semigroup – the right shift semigroup on $L^\infty(\mathbb{R})$ – is not strongly continuous.

An extensive theory of the duals of C_0-semigroups is available and is presented in the monograph [116].

Example 2.7 (Semigroups on $\ell^\infty(I;X)$)**.** Let $T = (T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Let I be a non-empty set and let $\ell^\infty(I;X)$ denote the set of all bounded families $x = (x_i)_{i\in I}$ together with the sup norm

$$\|x\|_\infty := \sup_{i\in I} \|x_i\|_X .$$

Consider the semigroup $S = (S(t))_{t\in[0,\infty)}$ on $\ell^\infty(I;X)$ given by

$$\big(S(t)x\big)_i = T(t)x_i$$

for all $x \in \ell^\infty(I;X)$ and $i \in I$. It is not difficult to check that, if I is infinite, then S is a C_0-semigroup if and only if T is uniformly continuous. Hence, the strong continuity of T does not carry over to S, in general.

This construction occurs, for instance, in [61, Section 3] to show operator norm convergence in an approximation theorem for operator semigroups, but for the reason pointed out above the argument is only applicable in case that the generator is bounded.

Example 2.8 (Semigroups on ultrapowers)**.** This is a slightly more involved – and more useful – variation of Example 2.7. Again, let $T = (T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X and let I be a non-empty set. Consider a filter $\mathcal{F} \subseteq 2^I$ on I and recall that a family $x = (x_i)_{i\in I}$ in X is said to *converge* to a vector $x_0 \in X$ *along the filter* $\mathcal{F}$ if the set $\{i \in I \mid \|x_i - x_0\| < \varepsilon\}$ belongs to $\mathcal{F}$ for each $\varepsilon > 0$. For instance, if $I = \mathbb{N}$ and if $\mathcal{F}$ is the so-called *Fréchet filter* that consists of all co-finite subsets of $\mathbb{N}$, then convergence along $\mathcal{F}$ is precisely the usual notion of convergence of sequences.

Let $c_{0,\mathcal{F}}(I;X)$ denote the closed subspace of $\ell^\infty(I;X)$ that consists of those families that converge to 0 along $\mathcal{F}$. The quotient space

$$X^{\mathcal{F}} := \ell^\infty(I;X)/c_{0,\mathcal{F}}(I;X)$$

is the $\mathcal{F}$*-power of* X; we refer to such spaces as *filter powers* of E. For a bounded linear operator $R \in \mathcal{L}(X)$ the induced operator

$$\begin{aligned} \ell^\infty(I;X) &\to \ell^\infty(I;X) \\ (x_i)_{i\in I} &\mapsto (Rx_i)_{i\in I} \end{aligned}$$

leaves $c_{0,\mathcal{F}}(I;X)$ invariant and thus induces a bounded linear operator $R^{\mathcal{F}}$ on the filter power $X^{\mathcal{F}}$.

The main idea here that underlies this construction is to ignore "what happens at finite indices". Hence, when using filter products one will typically use a filter $\mathcal{F}$ that contains all co-finite subsets of the index set I – for instance, $\mathcal{F}$ could be, as indicated above, the Fréchet filter that consists of all co-finite subsets of I. Yet, for such $\mathcal{F}$ the filter power $X^{\mathcal{F}}$ does sometimes not behave very nicely since it destroys a lot of nice properties of X. For instance, if X is finite-dimensional and $\mathcal{F}$ is the Fréchet filter, then $X^{\mathcal{F}}$ will be a non-reflexive non-separable Banach space. For this reason, it is often better to choose $\mathcal{F}$ to be an ultrafilter that contains the Fréchet filter. In this case, one calls the filter power $X^{\mathcal{F}}$ an *ultrapower* of X.

Ultrapowers tend to be quite well-behaved. For instance, one has $X^{\mathcal{F}} = X$ for finite-dimensional X. For infinite-dimensional X, an ultrapower $X^{\mathcal{F}}$ will still be much larger than X, but the construction will preserve a number of nice geometric properties. For instance, if X is a Hilbert space, every ultrapower of X is again a Hilbert space. If X is an L^p-space, so is every ultrapower of X. And X is *super reflexive* if and only if every ultrapower of X is reflexive (equivalently, super reflexive). For an overview of ultrapowers of Banach spaces and the slightly more general notion of *ultraproducts* (where one starts with different Banach spaces X_i rather than a single one), we refer to the classical survey article by Heinrich [70]. Ultraproducts are a very common tool in the geometry of Banach spaces. In operator theory, lifting an operator R to an ultrapower is a very powerful tool to study the spectrum $\sigma(R)$; see for instance [112, Section V] and [103, Chapter 4] for details.

Those techniques have been adapted to positive C_0-semigroups. The main challenge here – which brings us back to the topic of this survey – is that for a C_0-semigroup $T = (T(t))_{t\geq 0}$ on a Banach space X the operator family $T^{\mathcal{F}} := (T(t)^{\mathcal{F}})_{t\in[0,\infty)}$ on an ultrapower (or on a more general filter power) $X^{\mathcal{F}}$ will, while it still satisfies the semigroup law, typically not be strongly continuous. To resolve this, one can either restrict the semigroup $T^{\mathcal{F}}$ to its space of strong continuity or one can instead choose to only lift the resolvent of the generator rather than the semigroup itself to $X^{\mathcal{F}}$ (which results in a so-called *pseudo-resolvent* on $X^{\mathcal{F}}$). These techniques are very powerful in the spectral theory of positive semigroups, see [106, Section C-III].

Slightly different applications of ultraproducts and ultrapowers in a semigroup context are, for instance, the proof of Fendler's semigroup version [58] of the Akcoglu–Sucheston dilation theorem on L^p and the proof of a special case of the classical Markov group conjecture about C_0-groups on ℓ^1 [63].

Example 2.9 (Implemented semigroups)**.** Let $S = (S(t))_{t\in[0,\infty)}$ as well as $T = (T(t))_{t\geq 0}$ be C_0-semigroups on a Banach space X. Then the *implemented semigroup* $R = (R(t))_{t\in[0,\infty)}$ is the semigroup on the operator space $\mathcal{L}(X)$ defined by

$$R(t)C = S(t)CT(t)$$

for all $C \in \mathcal{L}(X)$ and all $t \in [0,\infty)$. The semigroup R will typically not be strongly continuous unless S and T are uniformly continuous.

Such implemented semigroup occur, for instance, in the context of quantum mechanics where X is a Hilbert space, T is the solution (semi)group of a Schrödinger equation, and S is its adjoint semigroup. For a simple illustration of this in finite dimensions we refer, for instance, to [24, Definition 3.26]. If X is a Hilbert space and one restricts the semigroup R to the space $\mathcal{K}(X)$ of compact operators, it becomes strongly continuous again. The implemented semigroup R on $\mathcal{L}(X)$ is then the bidual of $R|_{\mathcal{K}(X)}$ and the dual semigroup of $R|_{\mathcal{K}(X)}$ acts on the space of trace class operators and describes – in the quantum mechanical context mentioned above – the evolution of the quantum system on mixed states.

3. Starter: semigroups in finite dimensions

To provide context for the discussion of semigroups that are not strongly continuous, we start with an outline of some main ideas from C_0-semigroup theory in Sections 3 and 4. Readers who are well-versed in C_0-semigroup theory can safely skip these sections.

In fact, we find it quite illuminating to start with the finite-dimensional case, which we do in the present Section 3. That means that we essentially discuss linear autonomous ODEs and matrix exponential functions in this section. Our exposition is unlikely to teach the reader something new about this classical topic, though. Our point is rather to represent this topic from a particular perspective that helps to understand the infinite-dimensional situation in Section 4.

3.1. The initial value problem.

The theory of one-parameter semigroups is mainly about solving linear autonomous differential equations, also referred to as *Cauchy problems*. In finite dimensions, such problems have the following form:

For a matrix $A \in \mathbb{C}^{n\times n}$ and an initial vector $x_0 \in \mathbb{C}^n$ we look for a (continuously) differentiable function $x : [0,\infty) \to \mathbb{C}^d$ satisfying the initial value problem

$$\begin{cases} \dot{x}(t) = Ax(t) & \text{for all } t \geq 0, \\ x(0) = x_0. \end{cases} \tag{3.1}$$

The solution x to this problem can be written down explicitly in terms of the *matrix exponential function*: it is given by

$$x(t) = e^{tA}x_0 \qquad \text{for all } t \geq 0,$$

where e^{tA} is defined as the exponential series $e^{tA} := \sum_{k=0}^{\infty} (tA)^k/k!$. This series converges absolutely in $\mathbb{C}^{n\times n}$.

Let us mention different approaches to show that this indeed gives the solution to (3.1): The most common approach is to prove directly – as in the scalar-valued case – that the matrix exponential function $t \mapsto e^{tA}$ is differentiable with derivative $\frac{\mathrm{d}}{\mathrm{d}t}e^{tA} = Ae^{tA}$. Hence, $x(t) = e^{tA}$ satisfies the differential equation in (3.1). The initial condition follows from the simple observation $e^{0A} = \mathrm{id}$. Alternatively, one can approach the situation from the general theory of (nonlinear) ODEs and use the Picard–Lindelöf iteration to compute the solution x of (3.1). The Picard–Lindelöf iterates turn out to be the partial sums of the series $\sum_{k=0}^{\infty} \frac{(tA)^k}{k!}x_0$.

3.2. **Three central objects: semigroup, generator, resolvent.** One commonly associates three different objects to the initial value problem (3.1) which are referred to as the *semigroup*, the *generator*, and the *resolvent*. In this subsection, we briefly motivate the meaning of these notions.

The semigroup. For a matrix $A \in \mathbb{C}^{n\times n}$, the mapping $t \mapsto e^{tA}$ satisfies the properties

$$e^{(s+t)A} = e^{sA}e^{tA} \quad \text{for all } s,t \geq 0 \qquad \text{and} \qquad e^{0A} = \mathrm{id}\,.$$

The right equality above is an immediate consequence of the definition of e^{tA} and the property on the left can be checked as in the scalar case by writing $e^{sA}e^{tA}$, which is a product of two series, as a convolution. Alternatively, the formula follows from the fact that, for each $x_0 \in \mathbb{C}^d$, the mapping $x\colon t \mapsto e^{tA}x_0$ solves the autonomous initial value problem (3.1).

The formula $e^{(s+t)A} = e^{sA}e^{tA}$ means that the mapping $t \mapsto e^{tA}$ is a semigroup homomorphism between the additive semigroup $[0,\infty)$ and the multiplicative semigroup $\mathbb{C}^{n\times n}$ (and in fact, it is even a monoid homomorphism as it maps the neutral element to the neutral element). In operator theory, it is common to abbreviate the word "semigroup homomorphism" in this context and to simply call the family $(e^{tA})_{t\geq 0}$ a *semigroup*.

The generator. This word refers to A itself. The mapping $t \mapsto e^{tA}$ is differentiable and its derivative at t is the matrix Ae^{tA} – which is precisely why the solution to the initial value problem (3.1) can be expressed by means of the semigroup $(e^{tA})_{t\geq 0}$. In particular, the derivative of $t \mapsto e^{tA}$ at the time $t = 0$ is the matrix A and it is common to express this property by saying that A *generates* or *is the generator of* the semigroup $(e^{tA})_{t\geq 0}$.

The resolvent. The complement $\rho(A) := \mathbb{C} \setminus \sigma(A)$ of the spectrum of A is called the *resolvent set* of A and the mapping

$$\begin{aligned} \mathcal{R}(\,\cdot\,, A)\colon \rho(A) &\to \mathbb{C}^{n\times n}, \\ \lambda &\mapsto \mathcal{R}(\lambda, A) := (\lambda\,\mathrm{id} - A)^{-1} \end{aligned}$$

is referred to as the *resolvent* of A. If $|\lambda| > \|A\|$ (or, more generally, if $|\lambda|$ is larger than the modulus of every eigenvalue of A), the resolvent at λ can be computed by the geometric series. This gives $\mathcal{R}(\lambda, A) = \sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}}$ and is, for $n \neq 1$, more commonly referred to as the *Neumann series representation* of the resolvent.

To understand the importance of the resolvent, it is very instructive to consider the Laplace transform: For a – say for the moment, finite-dimensional – normed space V over $\mathbb{C}$ and a measurable function $f : [0,\infty) \to V$ that satisfies the exponential estimate $\|f(t)\| \leq Me^{\omega t}$ for constants $M \geq 0$, $\omega \in \mathbb{R}$ and all $t \in [0,\infty)$, the *Laplace transform* of f is the mapping

$$\begin{aligned} \hat{f} : \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda > \omega\} &\to V, \\ \lambda &\mapsto \int_0^\infty e^{-\lambda t} f(t)\,\mathrm{d}t. \end{aligned}$$

If $x : [0,\infty) \to \mathbb{C}^n$ is the continuously differentiable function that solves the initial value problem (3.1), then its Laplace transform $\hat{x}$ is given by

$$\hat{x}(\lambda) = \mathcal{R}(\lambda, A)x_0 \tag{3.2}$$

for all $\lambda \in \mathbb{C}$ with sufficiently large real part. Here are two different ways to see this:

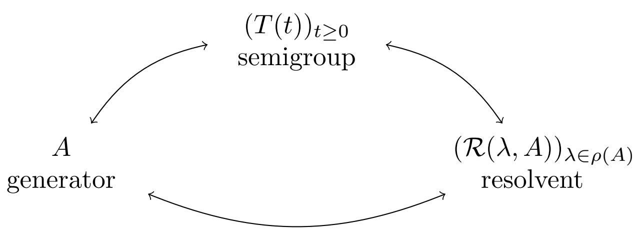
(1) We know the explicit solution formula $x(t) = e^{tA}x_0$ for all $t \in [0,\infty)$. The semigroup mapping $T : t \mapsto e^{tA}$ is exponentially bounded – it satisfies $\|e^{tA}\| \le e^{\|A\|t}$ for all $t \in [0,\infty)$ – and a short computations shows that its Laplace transform is, at all complex number λ with $\operatorname{Re}\lambda > \|A\|$, given by $\hat{T}(\lambda) = \mathcal{R}(\lambda, A)$. Multiplication with x_0 yields the claimed formula (3.2).
This approach is, strictly speaking, not really about the initial value problem (3.1). Rather, it directly relates the semigroup generated by A to the resolvent of A.

(2) The second approach uses a property of the Laplace transform, namely that $\widehat{\dot{x}}(\lambda)$ is equal to $\lambda\hat{x}(\lambda) - x(0)$. Due to this formula, taking the Laplace transform on both sides of the differential equation in (3.1) gives

$$\lambda\hat{x}(\lambda) - x_0 = A\hat{x}(\lambda)$$

and therefore $\hat{x}(\lambda) = \mathcal{R}(\lambda, A)x_0$ for all λ with sufficiently large real part.
This argument is mainly about the initial value problem (3.1) and did not involve the semigroup directly. Note however that we cheated a bit: to ensure that the argument actually works one first needs to show that the solution x is exponentially bounded. One way to do this without explicitly using the semigroup $(e^{tA})_{t\ge 0}$ is via Gronwall's lemma.

3.3. **Switching between semigroups, generators, and resolvents.** As in the previous Subsection 3.2 let us stay in the finite-dimensional setting and analyse how the three main objects in semigroup theory – the semigroup, the generator, and the resolvent – can be obtained from each other. In other words, we discuss the edges in the following graph that connects those three objects (see [48, p. 48] for the original picture):



While we formulate the connections in a finite-dimensional setting, they will guide us the way in the infinite-dimensional situation that we discuss from Subsection 4.1 on.

The semigroup and the generator. Consider a matrix $A \in \mathbb{C}^{n\times n}$ and the associated semigroup $(e^{tA})_{t\in[0,\infty)}$ in $\mathbb{C}^{n\times n}$. How to obtain both objects from each other?

If the semigroup is given, we have already seen in Subsection 3.2 how to obtain the generator A from it: it is the temporal of the semigroup at $t = 0$, i.e.,

$$A = \frac{\mathrm{d}}{\mathrm{d}t} e^{tA}|_{t=0}.$$

Conversely, assume now that the generator A is given. Here are two ways to get the semigroup operator e^{tA} from it at a given time $t \in [0,\infty)$:

(1) Via a *power series*: This is merely the definition of e^{tA} which we gave in Subsection 3.2. We defined

$$e^{tA} := \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}.$$

(2) Via *Euler's formula*: As for the scalar exponential function, one has the formula
$$e^{tA} = \lim_{n\to\infty} \left(\mathrm{id} + \frac{tA}{n}\right)^n .$$

We will see in Section 4 that both those formulas for e^{tA} cause considerable problems in the infinite-dimensional setting. This turns out to be different for formulas that rely on the resolvent rather than on the operator A itself. We discuss such formulas next.

The semigroup and the resolvent. For a matrix $A \in \mathbb{C}^{n\times n}$, how to obtain the semigroup $(e^{tA})_{t\ge 0}$ and the resolvent $\mathcal{R}(\,\cdot\,, A)$ from each other, without making explicit use of the generator A itself?

If the semigroup is given, then we already know from Subsection 3.2 that the resolvent can be obtained by computing the Laplace transform, i.e., one has
$$\mathcal{R}(\lambda, A) = \int_0^\infty e^{-t\lambda} e^{tA} \,\mathrm{d}t$$
whenever $\operatorname{Re}\lambda > \|A\|$. In fact, it is not difficult to show that the integral converges (absolutely) and coincides with $\mathcal{R}(\lambda, A)$ even if $\operatorname{Re}\lambda$ is only assumed to be strictly larger than the *spectral bound* $s(A)$, which is defined as the largest among all real parts of the eigenvalues of A. At points $\lambda \in \rho(A)$ that are located further to the left, the resolvent cannot be represented as a Laplace transform. However, it is worth noting that the values of the resolvent on any right half plane (or, more generally, on any non-empty open set) determine the resolvent everywhere else since $\mathcal{R}(\,\cdot\,, A)$ can be shown to be an analytic (complex analytic, holomoprhic) function.

Conversely, assume now that we have knowledge of $\mathcal{R}(\,\cdot\,, A)$ and want to obtain the semigroup at a time $t \in [0, \infty)$. We explain three ways to do this:

(1) Via *Euler's formula with negative exponents.* We explained above how to obtain e^{tA} from A via Euler's formula. First observe that for the scalar-valued case, i.e., the case $A \in \mathbb{C}$, one can instead make use of the Euler formula for e^{-tA}, which gives the formula $e^{tA} = (e^{-tA})^{-1} = \lim_{n\to\infty} \Big(1 - \frac{tA}{n}\Big)^{-n}$. The same can be shown to be true for $A \in \mathbb{C}^{n\times n}$, i.e., one has
$$e^{tA} = \lim_{n\to\infty} \left(\mathrm{id} - \frac{tA}{n}\right)^{-n} = \lim_{n\to\infty} \left(\frac{n}{t}\mathcal{R}\left(\frac{n}{t}, A\right)\right)^n .$$
This is actually a special case of the Post–Widder inversion formula for the Laplace transform, which can for instance be found in [10, Theorem 1.7.7] (for the scalar-valued case; by componentwise application, it is also true for matrix-valued functions). To see that the resolvent does indeed occur in the Post–Widder inversion formula, one needs to observe that the k-th derivative of the resolvent at a point $\lambda \in \rho(A)$ is given by $\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\mathcal{R}(\lambda, A) = (-1)^k \mathcal{R}(\lambda, A)^{k+1}$ for every integer $k \ge 0$.

(2) Via *Cauchy's integral formula*: For a closed path γ in the complex plane that encircles each spectral value of A exactly once, one has the following matrix-valued equivalent of Cauchy's integral formula:
$$e^{tA} = \frac{1}{2\pi \mathrm{i}} \oint_\gamma e^{t\lambda}\, \mathcal{R}(\lambda, A) \,\mathrm{d}\lambda.$$
This is a special case of the holomorphic functional calculus for matrices.

(3) Via *complex inversion of the Laplace transform*: As $\mathcal{R}(\,\cdot\,, A)$ is the Laplace transform of the semigroup, one can use the complex inversion formula for

the Laplace transform to get the semigroup from $\mathcal{R}(\,\cdot\,, A)$. This gives the formula

$$e^{tA} = \lim_{\sigma\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\omega-\mathrm{i}\sigma}^{\omega+\mathrm{i}\sigma} e^{t\lambda}\, \mathcal{R}(\lambda, A)\, \mathrm{d}\lambda$$

for any real number $\omega > s(A)$. Note that this is a version of Cauchy's integral formula in the previous point: If one closes the line from $\omega - \mathrm{i}\sigma$ to $\omega + \mathrm{i}\sigma$ by a half circle one gets, for large ω, a path that encircles the spectrum of A. But the integral over the half circle is negligible if σ is large; this can be shown by using the decay $e^{t\lambda} \to 0$ as $\operatorname{Re}\lambda \to -\infty$ together with the estimate $\|\mathcal{R}(\lambda, A)\| \le \frac{1}{|\lambda|-\|A\|}$ for $|\lambda| > \|A\|$ that follows from the Neumann series.

The generator and the resolvent. For a matrix $A \in \mathbb{C}^{n\times n}$ and a complex number λ in the resolvent set $\rho(A)$ the resolvent $\mathcal{R}(\lambda, A)$ is obtained from A plainly by its definition $\mathcal{R}(\lambda, A) = (\lambda\,\mathrm{id} - A)^{-1}$. Conversely, let us mention three different ways to retrieve A if the resolvent $\mathcal{R}(\,\cdot\,, A)$ is given:

(1) Via *reversing the definition*: This is most straightforward – for $\lambda \in \rho(A)$ one has $A = \lambda\,\mathrm{id} - \mathcal{R}(\lambda, A)^{-1}$.

(2) Via *Cauchy's integral formula*: Just as one can obtain e^{tA} from the resolvent by a Cauchy integral, the same also works for A itself: one has

$$A = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \lambda\, \mathcal{R}(\lambda, A)\, \mathrm{d}\lambda,$$

where γ is any path in the complex plane that encircles each spectral value of A exactly once.

(3) Via the *behaviour at infinity*: One has

$$A = \lim_{|\lambda|\to\infty} A\lambda\mathcal{R}(\lambda, A) = \lim_{|\lambda|\to\infty} \lambda\big(\lambda\mathcal{R}(\lambda, A) - \mathrm{id}\big).$$

Note that only the second part of the formula expresses A purely in terms of the resolvent, without referring back to A itself.

To see that the formula holds, first observe that the Neumann series gives $\mathcal{R}(\lambda, A) \to 0$ as $|\lambda| \to \infty$. Hence, $\lambda\mathcal{R}(\lambda, A) = \mathrm{id} + A\mathcal{R}(\lambda, A) \to \mathrm{id}$ as $|\lambda| \to \infty$. Multiplying with A gives the claim.

4. A taste of C_0-semigroups

In this section we give a brief outline of some central concepts of C_0-semigroup theory. The main motivation for the theory is again to study linear autonomous initial value problems, now with values in a Banach space. The relation between the semigroup and the initial value problem is now a bit subtler than in the finite-dimensional case, so we postpone its detailed discussion to Subsection 4.4. Before that we take a close look at the three central objects that already occurred in the finite-dimensional case: the semigroup, its generator, and the resolvent.

A major distinction compared to the finite-dimensional case is that the generator A will, in general, be a unbounded operator that is only defined on a subspace of a surrounding Banach space. This reflects the fact that in concrete PDE examples on, say, L^p-spaces, the generator is typically a differential operator which is only defined on a Sobolev space rather than on the entire L^p-space.

A second distinction is that, once one has made the step to unbounded operators, it is far from obvious which such operators occur as the generator of a C_0-semigroup. This is a central question in C_0-semigroup theory since it is closely connected to the well-posedness of initial value problems (Section 4.4) and it is answered by a couple of so-called *generation theorems* (Section 4.5).

Finally, Section 4.6 roughly outlines how (generators of) C_0-semigroups behave under perturbation.

4.1. **What is a C_0-semigroup?** Let A be a bounded operator on a Banach space X. Since for each $t \geq 0$,

$$\sum_{k=0}^{\infty} \frac{t^k \|A\|^k}{k!} = e^{t\|A\|},$$

the series $\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$ is absolutely convergent in $\mathcal{L}(X)$. Therefore, the exponential series $e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$ is well-defined in $\mathcal{L}(X)$. Just like in the finite-dimensional case, the mapping $t \mapsto e^{tA}$ satisfies the properties

$$e^{0A} = \mathrm{id} \qquad \text{and} \qquad e^{(s+t)A} = e^{sA}e^{tA} \quad \text{for all } s,t \geq 0.$$

Moreover, the mapping $[0,\infty) \ni t \mapsto e^{tA} \in \mathcal{L}(X)$ is continuous; the proofs are standard and can be found, for instance, in [21, Proposition 9.2]. Motivated by this, we give the definition of a C_0-semigroup.

Definition 4.1. A family of bounded operators $(T(t))_{t\geq 0}$ on a Banach space X is called a *semigroup* if it satisfies the functional equation

$$T(0) = \mathrm{id} \qquad \text{and} \qquad T(t+s) = T(t)T(s) \quad \text{for all } t \geq 0.$$

A semigroup $(T(t))_{t\geq 0}$ is said to be *strongly continuous* (or *C_0-semigroup*) if for each $x \in X$, the orbit map $t \mapsto T(t)x$ is continuous from $[0,\infty)$ to X.

It turns out that the algebraic semigroup property and the analytical strong continuity property mesh well. For instance, a semigroup of bounded operators $(T(t))_{t\geq 0}$ on a Banach space X is automatically a C_0-semigroup if it is strongly continuous at 0, i.e., $\lim_{t\downarrow 0} T(t)x = x$ for each $x \in X$; see [48, Proposition I.5.3]. In fact, using the weak instead of the strong operator topology turns out to be sufficient [48, Theorem I.5.8]:

Proposition 4.2. *A semigroup of bounded operators $(T(t))_{t\geq 0}$ on a Banach space X is strongly continuous if and only if it is weakly continuous, i.e., for each $x \in X$ and $x' \in X'$, the map $[0,\infty) \ni t \mapsto \langle x', T(t)x\rangle$ is continuous.*

As noted in Example 2.6, the dual family $(T(t)')_{t\geq 0}$ corresponding to a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X is again a semigroup on X', but need not be strongly continuous, in general. However, one can readily check that strong continuity of $(T(t))_{t\geq 0}$ always implies weak*-continuity of $(T(t)')_{t\geq 0}$ for a general C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Hence, if X is reflexive, then it follows from Proposition 4.2 that the dual semigroup of a C_0-semigroup is also a C_0-semigroup.

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Then for each $x \in X$, the set

$$\{T(t)x : t \in [0,1]\}$$

is a continuous image of a compact set, hence bounded. By the uniform boundedness principle, it follows that $(T(t))_{t\geq 0}$ is locally bounded, i.e., bounded on compact intervals. This observation yields a quite useful property of C_0-semigroups that is they are exponentially bounded [48, Proposition I.5.5]:

Proposition 4.3. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Then there exists $\omega \in \mathbb{R}$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.*

The *growth bound* of a C_0-semigroup $(T(t))_{t\geq 0}$ is defined as

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} \mid \exists\, M \geq 1\ \forall\, t \geq 0\ \ \|T(t)\| \leq Me^{\omega t}\} \in [-\infty, \infty).$$

If there exists a number $M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \geq 0$, then the semigroup is called *bounded*.

4.2. **The generator of a C_0-semigroup.** If A is a matrix, we have seen that the matrix exponential $t \mapsto e^{tA}$ is differentiable with derivative $\frac{d}{dt}e^{tA} = Ae^{tA}$. More generally – due to uniform convergence – the same property holds if A is a bounded operator on a Banach space X. Consequently, there is a one-to-one correspondence between the semigroup $(e^{tA})_{t\geq 0}$ and the initial value problem (3.1). Therefore, to relate a general C_0-semigroup $(T(t))_{t\geq 0}$ to the solution of an initial value problem, one gets interested in the differentiability of the orbit map $\xi_x := t \mapsto T(t)x$ for fixed $x \in X$. It turns out [21, Lemma 9.5] that ξ_x is differentiable on $[0, \infty)$ if and only if it is right differentiable at 0, and in this case

$$\dot{\xi}_x(t) = T(t)\dot{\xi}_x(0), \qquad t \geq 0.$$

Limiting ourselves to the subspace where the semigroup is right differentiable, yields the following notion:

Definition 4.4. The *generator* of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X is the linear operator defined as

$$\operatorname{dom}(A) := \left\{ x \in X \mid \lim_{h\downarrow 0} \frac{T(h)x - x}{h} \text{ exists} \right\}$$
$$Ax := \dot{\xi}_x(0) = \lim_{h\downarrow 0} \frac{T(h)x - x}{h}.$$

The generator of a C_0-semigroup is a closed and densely defined operator and determines the semigroup uniquely [48, Theorem II.1.4]. According to the closed graph theorem, $\operatorname{dom}(A) = X$ if and only if A is bounded and in this case, the semigroup is given by $T(t) = e^{tA}$. For this reason, it is common in some parts of the literature to use the notation $(e^{tA})_{t\geq 0}$ for a general C_0-semigroup generated by A.

Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Since existence of a strong limit implies existence of a weak limit, the operator

$$A_w x := w\text{-}\lim_{h\downarrow 0} \frac{T(h)x - x}{h}$$

extends A and is called the weak-generator of A; here $w\text{-}\lim$ denotes the limit in the weak topology of X. Similarly as strong continuity of a semigroup is equivalent to weak continuity (Proposition 4.2), one also has the following; see also [109, Theorem 2.1.3] and [111, Exercise 1.2.4].

Proposition 4.5. *The strong and weak generators of a C_0-semigroup on a Banach space coincide.*

The generator of a C_0-semigroup has quite useful properties. In particular, we have a generalization of the fundamental theorem of calculus:

Proposition 4.6. *Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. The following properties hold:*

(a) *The semigroup leaves* $\operatorname{dom}(A)$ *invariant, i.e.,* $T(t)\operatorname{dom}(A) \subseteq \operatorname{dom}(A)$ *for all $t \geq 0$. Furthermore,*
$$\dot{T}(t)x = T(t)Ax = AT(t)x \qquad (x \in \operatorname{dom}(A)).$$

(b) *For each $t \geq 0$ and $x \in X$, we have*
$$\int_0^t T(s)x \,\mathrm{d}s \in \operatorname{dom}(A) \quad \textit{and} \quad A\int_0^t T(s)x \,\mathrm{d}s = T(t)x - x.$$
In addition, if $x \in \operatorname{dom}(A)$*, then*
$$\int_0^t T(s)Ax \,\mathrm{d}s = T(t)x - x.$$

for all $t \geq 0$.

4.3. **The resolvent of a C_0-semigroup.** Since the generator of a C_0-semigroup is always closed, it is natural to study its spectral properties. Let us recall that for a closed operator A on a Banach space X, its *resolvent set* is defined as the set

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is bijective from } \operatorname{dom}(A) \text{ to } X\}$$

and the complement set $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is called the *spectrum* of A. By the closed graph theorem, the operator $\mathcal{R}(\lambda, A) := (\lambda - A)^{-1} \in \mathcal{L}(X, \operatorname{dom}(A))$ for each $\lambda \in \rho(A)$ is called the *resolvent of A at λ*. Another object of interest is the *spectral bound* defined as

$$s(A) := \sup\{\operatorname{Re}\lambda \in \sigma(A)\} \in [-\infty, \infty].$$

The resolvent of A is a *pseudo-resolvent*, i.e., it satisfies the *resolvent identity*

$$\mathcal{R}(\lambda, A) - \mathcal{R}(\mu, A) = (\mu - \lambda)\mathcal{R}(\lambda, A)\mathcal{R}(\mu, A)$$

for all $\lambda, \mu \in \rho(A)$. The analytic properties of the resolvent are summarized in the following proposition.

Proposition 4.7. *The following properties hold for a closed operator A on a Banach space X.*

(a) *The resolvent set $\rho(A)$ is open in $\mathbb{C}$. More precisely, if $\lambda \in \rho(A)$ and $\mu \in \mathbb{C}$ satisfy $|\mu - \lambda| < \|\mathcal{R}(\lambda, A)\|^{-1}$, then μ is also in $\rho(A)$ and the resolvent is given by the Taylor series representation*

$$\mathcal{R}(\lambda, A) = \sum_{k=0}^{\infty} (\mu - \lambda)^n \mathcal{R}(\lambda, A)^{k+1},$$

which converges absolutely in $\mathcal{L}(X)$.

(b) *The resolvent map $\mathcal{R}(\,\cdot\,, A)$ is complex differentiable on $\rho(A)$ with*

$$\frac{d^n}{d\lambda^n}\mathcal{R}(\lambda, A) = (-1)^n n! \mathcal{R}(\lambda, A)^{n+1}.$$

for all $n \in \mathbb{N}_0$.

For the spectral theory of closed operators on a Banach space, we refer to [124, Chapter VIII] and [48, Section IV.1]. While the resolvent set of a closed operator may be empty, the situation is vastly better for semigroup generators. For them, not only is the resolvent set non-empty but because $\omega_0(T) < \infty$, we even have Laplace transform representation of the resolvent:

Proposition 4.8. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A. Then*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega_0(T)\} \subseteq \rho(A)$$

and for each $x \in X$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0(T)$, we have

$$\mathcal{R}(\lambda, A) = \int_0^{\infty} e^{-\lambda s} T(s)x \, \mathrm{d}s;$$

where the integral converges both as an improper Riemann integral and as a Bochner integral. In particular, $s(A) \leq \omega_0(T)$.

Furthermore, if $M \geq 1$ and $\omega \in \mathbb{R}$ are such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, then $\|\mathcal{R}(\lambda, A)^k\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^k}$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$.

While the spectral bound is always dominated by the growth bound and equality always holds in finite dimension [21, Corollary 4.8], the equality may fail in infinite-dimensions [10, Example 5.1.11]. The cases where equality holds are rather important as in these cases one can assert from $s(A) < 0$ that the semigroup operators converge to 0 in the operator norm as $t \to \infty$. We refer to [13, Section 6.1] and the references therein for further insight into *spectral bound equals growth bound* condition.

4.4. **Connection to initial value problems.** We are finally ready to discuss the relationship between C_0-semigroups and the solutions of *abstract Cauchy problem* of the form

$$\begin{cases} \dot{x}(t) = Ax(t) & \text{for all } t \geq 0, \\ x(0) = x_0 \end{cases} \tag{4.1}$$

for an operator A on a Banach space X. We first need to define what we mean by a *solution*.

Definition 4.9. Let A be an operator on a Banach space X.

(a) A *classical solution* of the abstract Cauchy problem (4.1) is a continuously differentiable function $x : [0, \infty) \to X$ which satisfies the following properties: $x(0) = x_0$ and for all $t \geq 0$ one has $x(t) \in \mathrm{dom}(A)$ and $\dot{x}(t) = Ax(t)$.

(b) A *mild solution* of the abstract Cauchy problem (4.1) is a continuous function $x : [0, \infty) \to X$ which satisfies the following properties: for each $t \geq 0$ one has $\int_0^t x(s)\, \mathrm{d}s \in \mathrm{dom}(A)$ and

$$A \int_0^t x(s)\, \mathrm{d}s = x(t) - x_0.$$

Using the properties of C_0-semigroups stated in Proposition 4.7, we immediately see that C_0-semigroups yields both classical (for $x \in \mathrm{dom}(A)$) and mild (for $x \in X$) solutions of (4.1). Actually, we even have uniqueness [48, Propositions II.6.2 and II.6.4].

Proposition 4.10. *Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. For each $x_0 \in X$, the orbit map*

$$x(\,\cdot\,, x_0) := t \mapsto T(t)x_0$$

is the unique mild solution of (4.1)*. Moreover, if $x_0 \in \mathrm{dom}(A)$, then $x(\,\cdot\,, x_0)$ is the unique classical solution of* (4.1)*.*

For initial value problems, the existence, uniqueness, and continuous dependence on initial data is what is referred to as well-posedness. There are various definitions for the notion of well-posedness of (4.1), see the discussion of [48, Page 151–152]. We use the definition given in [48, Definition II.6.8].

Definition 4.11. For a closed operator A on a Banach space X, the abstract Cauchy problem (4.1) is said to be *well-posed* if the following conditions hold:

(a) The operator A is densely defined.

(b) For each $x_0 \in \mathrm{dom}(A)$, there exists a unique solution $x(\,\cdot\,, x_0)$ of (4.1).

(c) For every null sequence (x_n) in $\mathrm{dom}(A)$, we have $x(t, x_n) \to 0$ as $n \to \infty$ uniformly on compact subsets of $[0, \infty)$.

The choice of the above definition is justified by the following result [48, Theorem II.6.7] which relates C_0-semigroups to well-posed abstract Cauchy problems.

Theorem 4.12. *For a closed operator A on a Banach space X, the following are equivalent.*

(i) *The operator A generates a C_0-semigroup.*

(ii) *The abstract Cauchy problem* (4.1) *is well-posed.*

(iii) *The condition* (b) *in Definition 4.11 holds and $\rho(A) \neq \varnothing$.*

(iv) *The condition* (b) *in Definition 4.11 holds and there exists a sequence $\lambda_n \uparrow \infty$ such that $(\lambda_n - A)$ is surjective for all $n \in \mathbb{N}$.*

4.5. **Generation theorems.** A key problem of semigroup theory is to characterize linear operators that generates a semigroup.

Characterization in terms of the resolvent. In general, closed densely defined operator whose spectrum lies in a left-half plane need not generate a C_0-semigroup [48, Example II.3.2]. However, an additional growth estimate of the resolvent powers on a right half plane (Proposition 4.8) does turn to be sufficient. This is the celebrated Hille–Yosida theorem below that was proved independently E. Hille and K. Yosida.

Theorem 4.13 (Hille-Yosida theorem). *Let A be linear operator on a Banach space X and let $M \geq 1$ and $\omega \in \mathbb{R}$. The following are equivalent.*

(i) *The operator A is the generator of C_0-semigroup $(T(t))_{t\geq 0}$ that satisfies $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.*
(ii) *The operator A is closed, densely defined, $(\omega, \infty) \subseteq \rho(A)$, and the estimates $\left\|(\lambda-\omega)^k \mathcal{R}(\lambda, A)^k\right\| \leq M$ hold for all $\lambda > \omega$ and $k \in \mathbb{N}$.*

In particular, the generators of contraction semigroups can be characterized as follows:

Corollary 4.14. *The following are equivalent for a linear operator A on a Banach space X.*

(i) *The operator A generates a contraction C_0-semigroup $(T(t))_{t\geq 0}$ on X.*
(ii) *The operator A is closed, densely defined, and $(0, \infty) \subseteq \rho(A)$ and the estimates $\|\lambda\mathcal{R}(\lambda, A)\| \leq 1$ hold for all $\lambda > 0$.*

The proof of the Hille-Yosida theorem is quite involved and the details become much less technical for the contraction case. In fact, the Theorem 4.13 can even be deduced from Corollary 4.14 and that is how it is usually done in the literature, for instance in [48, Section II.3]. We find it instructive to outline the proof of Corollary 4.14: The implication "(i) ⇒ (ii)" is known from Proposition 4.8. For the implication, "(ii) ⇒ (i)", one defines the *Yosida approximants*

$$A_n := nA\mathcal{R}(n, A) \qquad n \in \mathbb{N},$$

which are bounded operators on X. Consequently, they generate C_0-semigroups $(e^{tA_n})_{t\geq 0}$. In fact, the resolvent bound in (ii) even implies that the semigroups $(e^{tA_n})_{t\geq 0}$ are contractive. It is then shown that for each $t \geq 0$, the sequence (e^{tA_n}) converges strongly as on $\operatorname{dom}(A)$, hence on X. The strong limit

$$T(t) := \lim_{n\to\infty} e^{tA_n} \qquad t \geq 0$$

is then shown to be a C_0-semigroup and finally that its generator is indeed A.

Characterization in terms of the generator. Until now, the characterizations of semigroup generators that we have seen require knowledge of the resolvent. However, Lumer and Phillips obtained conditions for closed operators to generate a contraction semigroup without any information on the spectrum. Recall that an operator A on a Banach space X is said to be *dissipative* if

$$\|(\lambda - A)x\| \geq \lambda \|x\| \qquad \text{for all } \lambda > 0 \text{ and } x \in \operatorname{dom}(A).$$

Theorem 4.15 (Lumer-Phillips theorem). *The following are equivalent for a densely defined dissipative operator A on a Banach space X.*

(i) *The closure of A generates a contraction semigroup on X.*
(ii) *The image $\operatorname{Rg}(\lambda - A)$ is dense in X for some (hence, all) $\lambda > 0$.*

For a densely defined operator A on a Banach space X, it turns out that if both A and its Banach space dual A' are dissipative, then $\operatorname{Rg}(1 - A)$ is dense in X [48, Corollary II.3.17] and hence by Lumer-Phillips theorem, $\bar{A}$ generates a contraction semigroup on X. On Hilbert spaces, the situation is much nicer. Indeed, if A is a densely defined operator on a Hilbert space H and A is skew-adjoint, i.e., $A^* = -A$, then it can be shown that A and $-A$ are both dissipative and closed [48, Proof of Theorem II.3.24]. Therefore, both A and $-A$ generate C_0-semigroups on H. In other words, A generates a group $(T(t))_{t\in\mathbb{R}}$ on H, i.e., the properties in Definition 4.1 hold when $[0,\infty]$ is replaced with $\mathbb{R}$. As A is skew-adjoint, the group is unitary, i.e., $T(t)^{-1} = T(t)^*$ for all $t \geq 0$. For general operators on Hilbert spaces, we have the following consequence of Lumer-Phillips theorem, see for instance [122, Lemma 3.17 and Theorem 3.16].

Corollary 4.16. *An operator A on a Hilbert space H generates a contraction C_0-semigroup on X if and only if $-A$ is m-accretive, i.e.,* $\operatorname{Re}\langle Ax\,,\,x\rangle \leq 0$ *for all* $x \in \operatorname{dom}(A)$ *and* $\operatorname{Rg}(\lambda - A) = H$ *for some* $\lambda > 0$.

Generation by sectorial operators. One disadvantage of the Hille-Yosida theorem is having to check to check boundedness estimates for all powers of the resolvent. This can be avoided if a boundedness estimate holds for the resolvent in some sector.

Definition 4.17. An operator A on a Banach space X is called *sectorial* if there exists $\omega \in \mathbb{R}, \theta \in (\pi/2, \pi)$ and $M > 0$ such that

$$S_{\theta,\omega} := \{\lambda \in \mathbb{C} \mid \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\} \subseteq \rho(A)$$

and

$$\|(\lambda - \omega)\mathcal{R}(\lambda, A)\| \leq M \quad (\lambda \in S_{\theta,\omega}).$$

Sectorial operators are always closed. Using functional calculus, it can be shown that sectorial operators always generate a semigroup. However, such semigroups need not be strongly continuous (see Section 5.2), as the operator A need not be densely defined. Imposing this missing property yields a generation result for sectorial operators [48, Theorem II.4.6].

Theorem 4.18. *The following are equivalent for an A on a Banach space X.*

(i) *The operator A generates a bounded C_0-semigroup $(T(t))_{t\geq 0}$ on X such that* $\operatorname{Rg} T(t) \subseteq \operatorname{dom}(A)$ *for all $t > 0$ and the family $(tAT(t))_{t\in[0,1]}$ is uniformly bounded.*
(ii) *The operator A is densely defined and sectorial.*

In fact, the C_0-semigroup $(T(t))_{t\geq 0}$ generated by densely defined sectorial operator has properties much stronger than the one stated in condition (i) above. Indeed, it is shown in [48, Theorem II.4.6] that $(T(t))_{t\geq 0}$ even extends to a family of bounded operators $(T(\lambda))_{\lambda\in S_{0,\theta-\pi/2}}$ such that

$$T(\lambda + \mu) = T(\lambda)T(\mu) \quad \text{for all } \lambda, \mu \in S_{0,\theta-\pi/2}$$

and the map $\lambda \mapsto T(\lambda)$ is analytic in the sector $S_{0,\theta-\pi/2}$. Furthermore, for each $\theta' \in (\pi/2, \theta)$ and $x \in X$, we have

$$\lim_{S_{0,\theta'-\pi/2}\ni\lambda\to 0} T(\lambda)x = x$$

The semigroups $(T(\lambda))_{\lambda\in S_{0,\theta-\pi/2}}$ above are called *analytic* and occur frequently in applications. This is specially because on Hilbert spaces, self-adjoint operators that are bounded above are sectorial and densely defined [48, Corollary II.4.7] and hence generate an analytic C_0-semigroup by Theorem 4.18. Generators of analytic semigroups exhibit remarkable spectral properties – for example, $s(A) = \omega_0(T)$ – that makes it easier to analyze the asymptotic behaviour of analytic semigroup.

Generation by form methods. In the realm of various evolution equations, the energy is often quantified through an L^2-norm, rendering Hilbert spaces a fitting framework for their formulation and analysis. Abstractly, the Riesz representation theorem, or more generally, the Lax-Milgram lemma, offers insights into the properties of existence and uniqueness, particularly suited for establishing weak solutions of elliptic partial differential equations.

Definition 4.19. Let V be a vector space. A *form* on V is a scalar-valued mapping on $V \times V$ that is sesquilinear, i.e., linear in the first and anti-linear in the second argument.

Moreover, we say that a is *bounded* if there exists $M \geq 0$ such that

$$|a(u,v)| \leq M \|u\| \|v\| \qquad \text{for all } u, v \in V,$$

and *coercive* if there exists $\alpha > 0$ such that

$$\operatorname{Re} a(u) \geq \alpha \|u\|^2 \qquad \text{for all } u \in V.$$

Let a be a bounded form on a Hilbert space V. Then $a(u, \cdot\,)$ is a bounded linear functional on V for each $u \in V$, so by the Riesz representation theorem, there exists a bounded linear operator $\mathcal{A}$ on V such that

$$\langle \mathcal{A}u \,,\, v\rangle = a(u,v)$$

for all $u, v \in V$. This operator is called the *Lax-Milgram operator (associated to a)*.

Theorem 4.20 (Lax-Milgram lemma)**.** *Let a be a bounded and α-coercive form on a Hilbert space V and let $\mathcal{A}$ denote the associated Lax-Milgram operator. Then $\mathcal{A}$ is an isomorphism and $\|\mathcal{A}^{-1}\| \leq \alpha^{-1}$.*

Definition 4.21. Let V and H be Hilbert spaces and let $j : V \to H$ be a dense embedding. Let $\mathcal{A}$ denote the Lax-Milgram operator associated to a bounded coercive form on V and define $k : H \to V'$ as $y \mapsto \langle y \,,\, j(\,\cdot\,)\rangle$. The operator A on H defined by

$$A^{-1} := j\mathcal{A}^{-1}k$$

is called the *operator associated with* (a, j).

Equivalently, one may define the operator A above as

$$A := \{(x,y) \in H \times H \mid \exists\, u \in V\ j(u) = x \text{ and } a(u, \cdot\,) = \langle y \,,\, j(\,\cdot\,)\rangle\};$$

see [122, Proposition 5.7]. In the situation of Definition 4.21, it can be shown using Lax-Milgram lemma that A is m-accretive [122, Theorem 5.6], so by Corollary 4.16 we have the following:

Theorem 4.22. *Let V and H be Hilbert spaces, let $j : V \to H$ be a dense embedding, and let a be a bounded coercive form on V. If A is the operator associated to (a, j), then $-A$ generates a contraction semigroup on H.*

Actually, it can even be shown [122, Exercise 5.2] that the semigroup $(T(t))_{t\geq 0}$ obtained in Theorem 4.22 is not only contractive but there even exists $\omega > 0$ such that $\|T(t)\| \leq e^{-\omega t}$ for all $t \geq 0$.

4.6. **Perturbation theorems.** A fundamental problem in the semigroup theory is to check whether perturbing the generator of a semigroup again yields a semigroup generator. To be more precise, if A generates a C_0-semigroup on a Banach space X and B is an operator on X, does $A + B$ also generate a C_0-semigroup on X? The problem is delicate because even if B is closed, it is possible that $A + B$ is not (take $B = -A$ for unbounded A). Moreover, it is also possible that $A + B$ is not densely defined. For bounded perturbations, the situation is rather nice. In this case, we do not only obtain that the perturbation generates a C_0-semigroup but we also have a representation formulae for the perturbed semigroup, see [48, Section III.1] for a proofs.

Proposition 4.23. *If A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X and $B \in \mathcal{L}(X)$, then $A+B$ also generates a C_0-semigroup $(S(t))_{t\geq 0}$ on X. Moreover, the following hold.*

(a) *The* variation of parameters *formula*
$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x$$
$$= T(t)x + \int_0^t S(s)BT(t-s)x$$
holds for all $t \geq 0$ and $x \in X$.

(b) *The perturbed semigroup is given by the* Dyson-Phillips series
$$S(t) = \sum_{n\geq 0} S_n(t)$$
where
$$S_0(t) = T(t) \qquad \text{and} \qquad S_{n+1}(t) = \int_0^t T(t-s)BS_n(s)\,\mathrm{d}s \quad (n \geq 0).$$

(c) *If $(T(t))_{t\geq 0}$ is analytic, so is $(S(t))_{t\geq 0}$.*

As pointed out above, when leaving the realm of bounded perturbations, it might not make sense to even ask where the perturbed operator still generates a C_0-semigroup. In particular, one might come across the following situations:

(1) The operator $A+B$ is not closed. This situation occurs for instance if A is unbounded and $B=-A$.
(2) The domain $D(A+B) = D(A) \cap D(B)$ is too small to be dense in X. For instance, if $D(A) \cap D(B) = \{0\}$.
(3) The resolvent set $\rho(A+B)$ is empty.

In order to tackle the second situation, one considers perturbations that are A-bounded:

Definition 4.24. Let A be an operator on a Banach space X. An operator B on X is said to be *A-bounded* if $\operatorname{dom}(A) \subseteq \operatorname{dom}(B)$ and there exists $a, b \geq 0$ such that
$$\|Bx\| \leq a\,\|Ax\| + b\,\|x\| \tag{4.2}$$
for all $x \in \operatorname{dom}(A)$. In this case,
$$a_0 := \inf\{a \geq 0 \,|\, \exists\, b \geq 0 \text{ such that (4.2) holds}\}$$
is called the *A-bound* of A.

To ensure that $A+B$ is a closed operator, it suffices to assume that B is A-bounded and $a_0 < 1$ [48, Lemma III.2.4]. Now, we have the following [48, Theorem 2.7]:

Proposition 4.25. *If A generates a contraction C_0-semigroup on a Banach space X and B is a dissipative and A-bounded operator on X with $a_0 < 1$, then $(A + B, \operatorname{dom}(A))$ also generates a contraction C_0-semigroup on X.*

In the situation of Proposition 4.25, if $a_0 \not< 1$ but (4.2) holds with $a = 1$, then the assertion remains true if, in addition, B' is densely defined on X', see [48, Corollary III.2.8]. Similar results can be found in [48, Section III.2]. These results are based on a series representation of the resolvent that prohibits one to estimate powers of the resolvent to apply Hille-Yosida theorem for non-contractive

C_0-semigroups. Instead approaching the problem based on the two variation of parameters formulae in Proposition 4.23(a) turns out to be more fruitful. In order to state the results, we first need to introduce the concept of interpolation and extrapolation semigroups.

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A and fix $\lambda \in \rho(A)$. The corresponding *interpolation space*

$$X_1 := (\operatorname{dom}(A), \|\cdot\|_1), \qquad \|\cdot\|_1 := \|A\cdot\|,$$

and *extrapolation space* – defined as the completion

$$X_{-1} := (X, \|\cdot\|_{-1})^{\sim}, \qquad \|\cdot\|_{-1} := \|\mathcal{R}(\lambda, A)\cdot\|,$$

are both Banach spaces. For different choices of λ, the norms on X_{-1} are equivalent. Further, $(T(t))_{t\geq 0}$ extends uniquely to a C_0-semigroup on X_{-1}, denoted by $(T_{-1}(t))_{t\geq 0}$. The generator A_{-1} of $(T_{-1}(t))_{t\geq 0}$ has domain $\operatorname{dom}(A_{-1}) = X$ and is the unique extension of A to a bounded operator from X to X_{-1}. Using X_1, one can construct the interpolation space X_2 continue the process and similarly for the extrapolation spaces. This yields the so-called *Sobolev tower* $(X_n)_{n\in\mathbb{Z}}$ and a C_0-semigroup on each of these spaces that has the same spectrum, spectral bound, and growth bound. A detailed description of interpolation and extrapolation spaces is given in [48, Section II.t].

We are now ready to state a *Desch-Schappacher perturbation* result given in [48, Corollary III.3.3].

Proposition 4.26. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X and let $B \in \mathcal{L}(X, X_{-1})$. Suppose that there exists $\tau > 0$ and $K \in (0, 1)$ such that*

(a) $\int_0^\tau T(\tau - s)Bu(s) \in X$ *and*
(b) $\left\|\int_0^\tau T(\tau - s)Bu(s)\right\|_X \leq K\,\|u\|_\infty$.

for all $u \in \mathrm{C}([0, \tau], X)$, then the part of $A_{-1} + B$ in X generates a C_0-semigroup given by the Dyson-Phillips series.

The above result is closely related to infinite-dimensional systems theory. Indeed, a control system can frequently be modelled as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad t \geq 0 \\ x(0) &= x_0; \end{aligned}$$

where A generates a C_0-semigroup on a Banach space X and $B \in \mathcal{L}(X, X_{-1})$. Then B satisfies the assumptions of Theorem 4.27 if it is *zero-class* C-*admissible*, i.e., for each $\tau > 0$, we have $\int_0^\tau T(\tau - s)Bu(s) \in X$ and

$$\lim_{\tau\downarrow 0} \left\|\int_0^\tau T(\tau - s)Bu(s)\right\|_X = 0$$

for all $u \in \mathrm{C}([0, \tau], X)$ In particular, zero-class C-admissible control operators are Desch-Schappacher perturbations. Other Desch-Schappacher perturbation results can be found in [48, Section III.3.a]. For the case of positive systems, these were recently treated in [20] and [14, Section 5]. The dual of control systems is an observation system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t), & t \geq 0 \\ y(t) &= Cx(t), & t \geq 0 \\ x(0) &= x_0; \end{aligned}$$

where A generates a C_0-semigroup on a Banach space X and $C \in \mathcal{L}(X_1, X)$. The observation operator C is called *zero-class L^1-admissible* if

$$\lim_{\tau \downarrow 0} \int_0^\tau \|CT(s)x\|_X \, \mathrm{d}s = 0$$

for all $x \in X_1$. These operators yield the so-called *Miyadera-Voigt perturbations*. More generally, we have

Proposition 4.27. *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X and let $C \in \mathcal{L}(X_1, X)$. Suppose that there exists $\tau > 0$ and $K \in (0,1)$ such that*

$$\int_0^\tau \|CT(s)x\|_X \, \mathrm{d}s \leq K \|x\|$$

for all $x \in X_1$, then $A + C$ generates a C_0-semigroup $(S(t))_{t\geq 0}$. Moreover,

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x$$

and

$$\int_0^\tau \|CD(s)x\|_X \, \mathrm{d}s \leq \frac{K}{1-K} \|x\|$$

for all $x \in X_1$ and $t \geq 0$.

For results regarding Miyadera-Voigt perturbations, we refer for instance to [48, Section III.3.c]. See also [110] and [68]. They play an important role in the study of Schrödinger and transport operators [118, 119]. Applications to systems theory can be found, for instance, in [69], [120, Section 3.2.1], and [14, Section 5]. Instead of additive, one can also consider multiplicative perturbations. We refer to [48, Section III.3.d] for an overview and [48, Notes to Chapter III] for further references.

5. Beyond the C_0-property

We now come to the heart of this article: semigroups which are not strongly continuous, but might have some other kind of time regularity property. The development of such theories typically follows two guiding principles:

(1) The theory is designed to solve a certain class of initial value problems.
(2) The theory gives meaning to appropriate version of the three important objects *semigroup*, *generator*, and *resolvent* and studies the relations between them. This encompasses, in particular, one or several generation theorems.

Any of this is only possible of the semigroup satisfies at least some kind of time regularity. If it does not, it will not be possible to take time derivatives of the orbits, to speak of a generator, or to use Laplace integrals. Hence, the different semigroup theories that we discuss in this section all require a non-trivial kind of time regularity and their distinctive feature is precisely what kind of regularity one assumes.

Yet, the rich variety of regularity notions (and hence of different semigroup theories) makes it most convenient to approach the subject from a very axiomatic point of view that defines a semigroup in a purely algebraic way and adds time regularity assumptions as and when needed.

5.1. **What is a semigroup?** Let $S = (S, +)$ be a commutative monoid and let $\leq$ be a pre-order on S given by

$$s \leq t \quad :\Leftrightarrow \quad \exists\, r \in S : t = s + r$$

for each $s, t \in S$. Since, $s, t \in s + t$ for all $s, t \in S$, so S is directed with respect to the pre-order $\leq$. A mapping $T : S \to \mathcal{L}(X)$ is called a *representation* of S on a Banach space X if it satisfies

$$T(0) = \mathrm{id} \quad \text{and} \quad T(t + s) = T(t)T(s) \quad \text{for all } t, s \in S.$$

With this notation, we call T an *operator semigroup* (over S on X) and typically use the notation $T = (T(s))_{s\in S}$ for it. The most common operator semigroup occurring in the literature are $(T(s))_{s\in[0,\infty)}$ or the discrete semigroup $(T^n)_{n\in\mathbb{N}}$ for a bounded operator $T \in \mathcal{L}(E)$. We limit ourselves to the former in this exposition and denote it by T, when there is no fear of confusion.

Definition 5.1. A Banach space valued function $f : [0, \infty) \to X$ is said to be *exponentially bounded* if there exists $M, \omega \in \mathbb{R}$ such that

$$\|f(t)\| \leq Me^{\omega t}$$

for all $t \geq 0$.

We know from Proposition 4.3 that a C_0-semigroup on a Banach space X is always exponentially bounded. When studying other operator semigroups, however, the assumption of exponential boundedness needs to be a priori assumed in order to obtain meaningful results.

5.2. **Regularity for $t > 0$.** The celebrated monograph [72] established the theory of operator semigroups that are merely strongly (or uniformly) continuous on $(0, \infty)$ treating C_0-semigroups as a special case. This theory was used to also study the n-parameter semigroup of linear operators [72, Section 10.10]. The choice of strong continuity was justified by the fact that it is already implied if one has strong mesurability on $(0, \infty)$ [72, Theorem 10.2.3]. In fact, strong mesurability on $(0, \infty)$ even implies boundedness near 0 [72, Lemma 10.2.1]. While most earlier results on such semigroups assume exponential boundedness, Baskakov in [19] developed a rather general theory for generators.

Let $(T(t))_{t>0}$ be a strongly continuous operator semigroup on a Banach space X. The *type* of the semigroup T defined as

$$\omega_0(T) := \lim_{t\to\infty} \frac{\ln \|T(t)\|}{t}$$

is always finite. The linear operator

$$\mathrm{dom}(A_0) := \left\{ x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}, \quad A_0x := \lim_{t\downarrow 0} \frac{T(t)x - x}{t}$$

is called the *infinitesimal operator* associated to the semigroup T. The infinitesimal operator need not be closed [19, Example 1] and in fact one even has $\rho(A_0) = \varnothing$ if $\overline{\operatorname{Rg} T} \neq X$ by [19, Corollary 3]. In particular, in the latter case, one cannot speak of the Laplace transform representation. For this reason, various other notions of generators were introduced in [19, Section 3]. The following one is particularly useful:

Definition 5.2. Let $(T(t))_{t>0}$ be a strongly continuous operator semigroup on a Banach space X. A linear relation $\mathcal{A}$ on X is said to be a *primitive generator* of T if it fulfills the following conditions:

(a) $\{(x, A_0x) \,|\, x \in \mathrm{dom}(A_0) \text{ and } \lim_{t\downarrow 0} T(t)A_0x = A_0x\} \subseteq \mathcal{A}$,

(b) $\mathrm{dom}(\mathcal{A}) \subseteq \overline{\operatorname{Rg} T}$,

$$T(t)x - T(s)x = \int_s^t T(\tau)y \, \mathrm{d}\tau \qquad (s \leq t)$$

for all $(x, y) \in \mathcal{A}$,

(c) $\mathcal{A}$ permutes with T, i.e., $(T(t)x, T(t)y) \in \mathcal{A}$ for all $t > 0$ and all $(x, y) \in \mathcal{A}$, and
(d) there exists $\omega_{\mathcal{A}} \geq \omega_0(T)$ such that

$$\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega_{\mathcal{A}}\} \subseteq \rho(A).$$

Primitive generators were also called *base generators* in [18]. A primitive generator is single-valued, i.e., a linear operator, if and only if $\ker T = \{0\}$; see [19, Corollary 2]. Sufficient conditions for existence of primitive generators are given in [19, Theorem 6 and Corollary 6]. For primitive generators the Laplace transform representation of the resolvent always holds in a certain subset [19, Theorem 4]:

Theorem 5.3. *Let $\mathcal{A}$ be a primitive generator of a strongly continuous operator semigroup $(T(t))_{t>0}$ on a Banach space X.*

Let $\lambda \in \mathbb{C}$ be such that $\operatorname{Re} \lambda > \omega_{\mathcal{A}}$ and let $x \in X$. If the conditions

$$\int_0^1 \|T(s)x\| \; \mathrm{d}s < \infty \quad \text{and} \quad \sup_{\tau \in (0,1]} \tau^{-1} \left\| \int_0^\tau T(s + \cdot)x \right\| \; \mathrm{d}s \in L^1[0, 1]$$

and

$$\int_0^1 \|T(s)\mathcal{R}(\lambda, \mathcal{A})x\| \; \mathrm{d}s < \infty \quad \text{and} \quad \lim_{\tau \downarrow 0} \tau^{-1} \int_0^\tau T(s)\mathcal{R}(\lambda, \mathcal{A})x \; \mathrm{d}s = \mathcal{R}(\lambda, \mathcal{A})x$$

are satisfied, then

$$\mathcal{R}(\lambda, \mathcal{A})x = -\int_0^\infty e^{-\lambda s} T(s)x \; \mathrm{d}s.$$

The situation improves if the semigroup is assumed to be exponentially bounded. First of all, let us point out boundedness near 0 implies exponential boundedness.

Proposition 5.4. *Let $(T(t))_{t>0}$ be a strongly continuous operator semigroup on a Banach space X. If there exists $M > 0$ such that $\sup_{t \leq 1} \|T(t)\| \leq M$, then T is exponentially bounded.*

Proof. For each $t \geq 0$, there exists $n \in \mathbb{N}$ and $s \in (0, 1]$ such that $t = s + n$. By the semigroup law,

$$\|T(t)\| = \|T(s)T(1)^n\| \leq M^{n+1} = Me^{n \log M} \leq Me^{\omega t}$$

for $\omega = \log M$. Hence, T is exponentially bounded. □

The most general semigroups for which ergodic theory was studied in [72, Section 18.4] are called *semigroups of class (E)*, see [72, Definition 18.4.1] for the definition. Semigroups of class (E) always have a primitive generator and one has the corresponding Laplace transform representation for the resolvent [19, Theorem 2]. The most well-studied class (E) semigroups are exponentially bounded semigroups. They actually belong to the larger class $(0,C_1)$ of semigroups, see [19, Section 1] for which we have the following due to [72, Corollary to Theorem 11.5.1 and Corollary 1 to Theorem 11.5.3] and [67, Corollary A.8.3]:

Theorem 5.5. *Let $(T(t))_{t>0}$ be a strongly continuous operator semigroup on a Banach space X. If T is exponentially bounded and*

$$\lim_{t \downarrow 0} \frac{1}{t} \int_0^t T(\tau)x \; \mathrm{d}\tau = x$$

for all $x \in X$, then its infinitesimal operator A_0 is closed and

$$\mathcal{R}(\lambda, A_0)x = \int_0^\infty e^{-\lambda s} T(x)x \; \mathrm{d}s$$

for all $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_0(T)$ *and all* $x \in X$*. Moreover, its* degeneration space

$$N_T := \{x \in X : T(t)x = 0 \text{ for all } t \geq 0\} = \{0\},$$

and the space of strong continuity of T *is* $\overline{\operatorname{dom}(A_0)}$*.*

The constant zero semigroup is a trivial example of a semigroup that shows that the condition

$$\lim_{t \downarrow 0} \frac{1}{t} \int_0^t T(\tau)x \, \mathrm{d}\tau = x \qquad (x \in X)$$

cannot be dropped in Theorem 5.5.

The semigroups occurring in parabolic problems can be analytically extended to a sector in the complex plane. Analytic semigroups can simply be constructed via the natural functional calculus and in turn, they exhibit particular strong properties. Extensive attention has been given to the study of analytic semigroups and we refer, in particular, to the self-contained monograph [96], the treatment in [67, Section 3.4], and [72, Chapter XVII]. We also refer to [11, 12, 91] for applications to second-order parabolic equations with non-local boundary conditions.

Let A be a *sectorial* operator on a Banach space X, i.e., there exists $\omega \in \mathbb{R}, \theta \in (\pi/2, \pi)$ and $M > 0$ such that

$$S_{\theta,\omega} := \{\lambda \in \mathbb{C} \mid \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\} \subseteq \rho(A)$$

and

$$\|(\lambda - \omega)\mathcal{R}(\lambda, A)\| \leq M \quad (\lambda \in S_{\theta,\omega}).$$

Then A is closed and there is a family $\{T(\lambda) : \lambda \in S_{0,\theta-\pi/2}\}$ of bounded linear operators such that

$$T(\lambda + \mu) = T(\lambda)T(\mu) \quad \text{ for all } \lambda, \mu \in S_{0,\theta-\pi/2}.$$

Moreover, the function $\lambda \mapsto T(\lambda)$ is analytic on $S_{0,\theta-\pi/2}$ and the representation

$$\mathcal{R}(\lambda, A) = \int_0^\infty e^{-\lambda s} T(s) \, \mathrm{d}s$$

holds for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$. In particular, $(T(t))_{t>0}$ is a strongly continuous bounded operator semigroup on X. Conversely, if $(T(t))_{t>0}$ is an exponentially bounded operator semigroup on X such that $t \mapsto T(t)$ is differentiable on $(0, \infty)$, the family $(tT'(t))_{t>0}$ is also exponentially bounded, and $T(t_0)$ is injective, then by [96, Propostion 2.1.9] there exists a sectorial operator on X whose resolvent is given by the Laplace transform representation of the semigroup (on a suitable half-plane).

5.3. **Bi-continuous semigroups.** In the context of one-parameter semigroups on Banach spaces we saw elementary constructions in Subsection 2.4 that also result in a semigroup, but possibly without the property of strong continuity. More generally, if a one-parameter semigroup is not strongly continuous with respect to the norm topology of a Banach space X, it may still have this property for some coarser locally convex topology τ on X. This is the case, e.g., for adjoint semigroups, for implemented semigroups or semigroups arising from product type constructions. Beside these, there are also natural examples that come from particular applications, such as transition semigroups of Markov processes or Koopman semigroups of dynamical systems (see Subsection 2.2).

In the literature there are many approaches that rectify the situation in very particular cases, and these are usually indeed attached to one specific class of problems. One of the right ideas, which is general enough to be applicable in many different situations, was discovered by Franziska Kühnemund and presented in her

PhD thesis [89], see also [90]. We describe here the setting and some basic results, then we give an overview of what is known for this class of semigroups.

The standing assumption in this subsection is the following:

Assumption 5.6. We consider a triple $(X, \|\cdot\|, \tau)$, where

(a) X is a Banach space with norm $\|\cdot\|$,
(b) τ is a Hausdorff locally convex topology, coarser than the norm-topology on X.
(c) τ is sequentially complete on norm-bounded sets, i.e., every $\|\cdot\|$-bounded τ-Cauchy sequence is τ-convergent.
(d) The continuous dual $(X,\tau)'$ space of (X,τ) is norming for X, i.e.,

$$\|x\| = \sup_{\substack{\varphi\in(X,\tau)' \\ \|\varphi\|\leq 1}} |\varphi(x)| \text{ for all } x \in X.$$

Note here that if the continuous dual $(X,\tau)'$ of (X,τ) coincides with the norm dual X' of X, then τ is finer than the weak topology $\sigma(X, X')$ and hence the semigroup is weakly continuous, hence in fact a C_0-semigroup on X by Theorem 4.2.

The advantage of keeping also the norm of the Banach space as the subject of study is that one can make simple but effective estimates, and, in particular, use spectral theory in the Banach algebra $\mathcal{L}(X)$ of bounded linear operators on X. The technical assumptions about the locally convex topology are useful for (1) evaluating Riemann integrals and (2) for making norm estimates, as we shall see shortly. We remark that the theory of Saks spaces, as described in ultimate detail in [36], is strongly connected to the above set of assumptions. We also note that the last condition (d) can be equivalently reformulated as the existence of a set $\mathcal{P}$ of τ-continuous seminorms defining the topology τ, such that $\|x\| = \sup_{p\in\mathcal{P}} p(x)$, see [30, Remark 4.2] and [82, Lemma 4.4].

Situations where these assumptions are fulfilled include the following: dual spaces X' with τ the weak* topology, space of bounded continuous functions (or some of it closed subspaces) over a compactly generated, completely regular space with τ the compact-open topology, and the Banach algebra of bounded linear operators over a Banach space with τ the strong operator topology.

Definition 5.7. Let $(X, \|\cdot\|, \tau)$ be a triple satisfying Assumption 5.6. We call a one-parameter operator semigroup $(T(t))_{t\geq 0}$ *τ-bi-continuous semigroup* if

(1) $(T(t))_{t\geq 0}$ is *τ-strongly continuous*, i.e., the map $\xi_x : [0,\infty) \to (X,\tau)$ defined by $\xi_x(t) = T(t)x$ is continuous for every $x \in X$.
(2) $(T(t))_{t\geq 0}$ is exponentially bounded., i.e., there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for each $t \geq 0$.
(3) For every $t_0 \geq 0$ and every norm-bounded τ-null sequence $(x_n)_{n\in\mathbb{N}}$ in X one has $T(t)x_n \xrightarrow{\tau} 0$ uniformly for $t \in [0, t_0]$.

Based on the exponential boundedness, the type (M,ω) and the growth bound $\omega_0(T)$ of a bi-continuous semigroup $(T(t))_{t\geq 0}$ can be defined as above, after Proposition 4.3. By virtue of Proposition 5.4 it is, of course, sufficient to assume opertor norm boundedness on any open interval $(0,a)$, $a > 0$.

Prominent examples include adjoints of C_0-semigroups, Koopman semigroups, Markov transition semigroups, and implemented semigroups. For more examples we refer to the literature detailed below, but note here explicitly that the shift semigroup, as a particular example of a Koopman semigroup, on the space of bounded continuous functions over $\mathbb{R}$ is bi-continuous with respect to the compact-open topology (but it is not a C_0-semigroup for the supremum norm, as has been discussed above, see Example 2.4).

The generator of bi-continuous semigroup is defined in complete analogy to the C_0-case:

Definition 5.8. Let $(T(t))_{t\geq 0}$ be a τ-bi-continuous semigroup over $(X, \|\cdot\|, \tau)$. The *generator* of $(T(t))_{t\geq 0}$ is the linear operator A defined by

$$Ax := \tau\lim_{t\to 0} \frac{T(t)x - x}{t}$$

with domain

$$\operatorname{dom}(A) := \Big\{ x \in X : \ \tau\lim_{t\to 0} \frac{T(t)x - x}{t} \text{ exists and } \sup_{t\in(0,1]} \frac{\|T(t)x - x\|}{t} < \infty \Big\}.$$

The most basic properties of these objects are collected in the following theorem:

Theorem 5.9. *Let $(T(t))_{t\geq 0}$ be a τ-bi-continuous semigroup with generator A.*

(a) *For each norm bounded sequence $(x_n)_{n\in\mathbb{N}}$ in* $\operatorname{dom}(A)$ *with $(Ax_n)_{n\in\mathbb{N}}$ norm bounded, $x_n \xrightarrow{\tau} x$ and $Ax_n \xrightarrow{\tau} y$ for some $x, y \in X$, one has $x \in \operatorname{dom}(A)$ and $Ax = y$. We call such an operator A* τ-bi-closed.

(b) *For each $x \in X$ there is a norm-bounded sequence $(x_n)_{n\in\mathbb{N}}$ in* $\operatorname{dom}(A)$ *such that $x_n \xrightarrow{\tau} x$. We say that* $\operatorname{dom}(A)$ *is* τ-bi-dense.

(c) $\operatorname{dom}(A)$ *is invariant under each $T(t)$ and $T(t)A = AT(t)$ for all $t \geq 0$.*

(d) *For $t > 0$ and $x \in X$ one has*

$$\int_0^t T(s)x \, ds \in \operatorname{dom}(A) \quad \textit{and} \quad A\int_0^t T(s)x \, ds = T(t)x - x,$$

where the integral has to be understood as a τ-Riemann integral (as explained above).

(e) *Each $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega_0(T)$ belongs to the resolvent set $\rho(A)$ (thus A is a closed operator) and, moreover, for each $x \in X$*

$$\mathcal{R}(\lambda, A)x = \int_0^\infty e^{-\lambda s} T(s)x \, ds.$$

Here the integral is the norm limit of the τ-Riemann integrals $\int_0^N$.

(f) *The space of strong continuity of $(T(t))_{t\geq 0}$ is* $\overline{\operatorname{dom}(A)}$.

Based on the above resolvent representation one can easily prove that a generator A of bi-continuous semigroup is a Hille-Yosida operator, i.e., satisfies the norm estimates of the resolvents as in (ii) of Theorem 4.13. It follows that on a reflexive Banach space the domain $\operatorname{dom}(A)$ of the generator of a bi-continuous semigroup is dense, so by virtue of the Hille-Yosida theorem the semigroup is already strongly continuous for the norm of the Banach space. In general, however $\operatorname{dom}(A)$ need not be norm dense, i.e., by the above the space of strong continuity is a proper subspace of X.

For the characterization of the generator it is useful to make the following definition:

Definition 5.10. A family $\mathcal{B} \subset \mathcal{L}(X)$ is called τ-bi-equicontinuous if for each norm bounded τ-null sequence $(x_n)_{n\in\mathbb{N}}$ one has $Bx_n \xrightarrow{\tau} 0$ uniformly for $B \in \mathcal{B}$.

The following analogue of the Hille-Yosida theorem was found by Kühnemund, see [90].

Theorem 5.11. *Let A be a linear operator on X. For $M \geq 1$ and $\omega \in \mathbb{R}$ the following assertions are equivalent:*

(i) *A generates τ-bi-continuous semigroup of (exponential) type (M, ω).*

(ii) *A is τ-bi-closed, τ-bi-densely defined with $(\omega,\infty)\subseteq\rho(A)$ and the family*
$$\{(\lambda-\omega)^n\mathcal{R}(\lambda,A)^n : \lambda>\omega, n\in\mathbb{N}\}$$
is a norm bounded and τ-bi-equicontinuous subset of $\mathcal{L}(X)$.

In her PhD thesis Kühnemund also worked out a substantial part of approximation theory for bi-continuous semigroups, including Trotter-Kato type theorems, the Lax-Chernoff product formula. In particular she showed the validity of Euler's formula (the convergence of the backward Euler scheme), in this context, called also the Post-Widder inversion: For each bi-continuous semigroup
$$T(t)x = \tau\lim_{m\to\infty}\left[\tfrac{m}{t}\mathcal{R}(\tfrac{m}{t},A)\right]^m x = \tau\lim_{m\to\infty}\left[\mathrm{id}-\left(\tfrac{t}{m}A\right)\right]^{-m}x \quad \text{for each } x\in X.$$
For a short proof relying on the C_0-semigroup theory we refer to [30], see also the more general paper by Cachia [35]. The picture of the basic theory of bi-continuous semigroups is hence complete: one has one-to-one correspondence between generators, resolvents and semigroups, as described above for the C_0-semigroups. In the last twenty years or so the abstract theory has been augmented as follows (we do not mention all the applications of the theory though):

Approximation. Large parts of approximation theory was worked out in the thesis of Kühnemund [89]. Cachia in [35] put Euler's formula for bi-continuous semigroups in a more general context. Trotter-Kato theorems were further studied by Albanese and Mangino in [3] with applications to Feller semigroups. Rational approximation schemes (beyond backward Euler) are covered in [74] by Jara.

Perturbation. Elements of perturbation theory are to be found in [53], [54] with applications in [51], [50]. Recent additions to perturbation theory are in the works of Budde, see [31], [25], including also positive Miyadera [27] and Desch-Schappacher [26] perturbations.

Extrapolation and interpolation spaces. Extrapolation spaces are constructed in [30]. For interpolation theory with applications (e.g., to non-linear Schrödinger equatios) we refer to the works [93] and [92] of Kunstmann.

Contraction semiroups. Lumer-Phillips type generation results were first obtained by Budde and Wegner in [34], then by Kruse and Seifert in [87].

Miscellany. Adjoint bi-continuous semigroups are studied in [55], see also the related papers by Kunze [94] and by Manca [99]. Bi-continuous cosine families were introduced by Budde in [29].

Mixed topologies and Saks spaces. The relation to classes of strongly continuous semigroups on locally convex spaces (such as Saks spaces, as mentioned above) is studied by Kraaij in [82], Kruse in [85], Kruse, Schwenninger in [86], see also [52,54].

Abstract Cauchy problems. Relation to solvability of Cauchy problems is studied in [89], [52], Domanski, Langenbruch [41] more recently in [84] by Kruse. The extension to control theory is due to Kruse and Seifert [88].

Asymptotic properties. Weak individual stability is studied in [46] but apart from this paper not much is known in theory of asymptotic properties of bi-continuous semigroups (for the case of semigroups on locally convex spaces we mention the paper by Jacob and Wegner [73]). A systematic study of stability properties of bi-continuous semigroups was initiated in [28]. Prior to that mean-ergodicity (i.e., Cesàro convergence) was studied by Albanese, Lorenzi, Manco in [2]. Concerning applications we mention the paper [39] by Dobrick, in which he studies asymptotic properties of flow semigroups on infinite networks, see also his PhD thesis [40].

Applications. We present here a very narrow selection of topics where bi-continuous semigroups can be useful (and only mention a few exemplary papers that are directly connected to the bi-continuous setting): parabolic and elliptic equations with possibly degenerate and unbounded coefficients and transition semigroups of Markov processes were studied by Farkas, Lorenzi [56], Es-Sarhir, Farkas [50, 51], Gerlach, Glück, Kunze, [60], Da Parto, Röckner [37], Kunze [95]. Koopman semigroups induced by jointly continuous semi-flows on compactly generated, completely regular spaces were studied by Dorroh, Neuberger [44, 45], Farkas, Kreidler [57]. Weighted and unweighted Koopman semigroups on spaces of holomorphic functions are discussed by Kruse [84] and flows on infinite networks by Budde, Fijavž [32].

5.4. **Integrated semigroups.** Let A be a densely defined operator on a Banach lattice E. If A is resolvent positive, i.e., there exists $\omega \in \mathbb{R}$ with $\omega > s(A)$ and $\mathcal{R}(\lambda, A) \geq 0$ for all $\lambda > \omega$ and the topological interior of E_+ is non-empty, then A is the generator of a (positive) C_0-semigroup on E [8, Corollary 2.3]. Other sufficient conditions for densely defined resolvent positive operators to be generate a (positive) C_0-semigroup were also studied by Arendt; see [8] or [6, Section II.2].

A natural way to construct resolvent positive operators that are not generators of C_0-semigroup is via perturbations. Indeed, on the Banach lattice $E = \{f \in C[0,1] : f(0) = 0\}$, the operator

$$\begin{aligned} \operatorname{dom}(A) &:= \{f \in C^1[0,1] : f'(0) = f(0) = 0\} \\ Af(x) &:= f'(x) + \frac{1}{2x} f(x) \qquad x \in (0,1] \end{aligned}$$

is resolvent positive, yet does not generate a C_0-semigroup [6, Example 3.2]. The first example of a resolvent positive operator that does not generate a C_0-semigroup was given by Batty and Davies in [22] and various other examples can be found in [6, Section II.3].

Nevertheless, resolvent positive operators generate the so-called *integrated semigroup*. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with generator A on a Banach space X. Integrating the Laplace transform representation of the resolvent and integrating by parts $k \in \mathbb{N}$ times, one gets

$$\mathcal{R}(\lambda, A) = \lambda^k \int_0^\infty e^{-\lambda t} S(t)\, dt \qquad \lambda > \omega_0(T);$$

where

$$S(t) := x \mapsto \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} T(s)x\, ds \qquad (t \geq 0).$$

In this case, $S : [0,\infty) \to \mathcal{L}(X)$ turns out to be strongly continuous function. This motivates the following definition:

Definition 5.12. For $k \in \mathbb{N}_0$, an operator A on a Banach space X is called the *generator of a k-times integrated semigroup* if there exists an exponentially bounded strongly continuous function $S : [0,\infty) \to \mathcal{L}(X)$ and $\omega \in \mathbb{R}$ such that

$$\mathcal{R}(\lambda, A) = \lambda^k \int_0^\infty e^{-\lambda t} S(t)\, dt \qquad \lambda > \omega.$$

In this case, we simply call S a *k-times integrated semigroup (generated by A).*

Of course, every 0-times integrated semigroup is a C_0-semigroup and if A generates a k-times integrated semigroup, then it also generates an n-times integrated semigroup for each $n > k$.

Remark 5.13. If A generates a once-integrated semigroup $(S(t))_{t\geq 0}$ on a Banach space X, then the part of A in $\overline{\operatorname{dom}(A)}$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on $\overline{\operatorname{dom}(A)}$ that satisfies

$$S(t)x = \int_0^t T(s)x \, dx \qquad (x \in \overline{\operatorname{dom}(A)});$$

see [9, Section 6].

If A is a resolvent positive operator on a Banach lattice E such that either A is densely defined or E has order continuous norm, then A is the generator of a once-integrated semigroup [6, Theorems 4.1 and 5.7]. Moreover, if A generates a C_0-semigroup on a Banach space X, then its Banach space dual A' generates a once-integrated semigroup on X' [10, Corollary 3.3.7].

The following is a generalization of the Hille-Yosida theorem for k-times integrated semigroups, see [10, Theorem 3.3.2] for a proof:

Theorem 5.14. *Let A be a densely defined linear operator on a Banach space X and let $M \geq 0, \omega \in \mathbb{R}$, and $k \in \mathbb{N}_0$. The following are equivalent.*

(i) *The operator A generates a k-times integrated semigroup $(S(t))_{t\geq 0}$ on X satisfying $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.*
(ii) *The interval (ω, ∞) lies in $\rho(A)$ and*

$$\sup_{n\in\mathbb{N}_0} \sup_{\lambda>\omega} \left\| \frac{(\lambda-\omega)^{n+1}\big(\lambda^{-k}\mathcal{R}(\lambda, A)\big)^{(n)}}{n!} \right\| \leq M.$$

Above, $\big(\lambda^{-k}\mathcal{R}(\lambda, A)\big)^{(n)}$ denotes the nth derivative of $\lambda \mapsto \lambda^{-k}\mathcal{R}(\lambda, A)$. A version of Theorem 5.14 for operators that are not necessarily densely defined can be found in [10, Theorem 3.3.1]. Various other generation theorems can be found in [10, Section 3.2] and [98, Section 3.6 and 3.7].

Integrated semigroups were introduced by Arendt in [6, 8, 9] and further developed by Neubrander [107], Kellermann [80], and Thieme [115]. For a systematic treatment of the theory, we refer to [10, Section 3.2] or [98, Chapter 3], the monographs [38, 123], and the survey [117]. The definition of integrated semigroups has also been extended to α-times integrated semigroups for $\alpha \geq 0$ by Hieber [71]. For ergodic and spectral theory of integrated semigroups, we refer to [16, 17, 113, 114] and the references therein. Application of integrated semigroups include – but are not limited to – population biology [98, Section 3.8], delay equations [1], and second order Cauchy problems [108].

Note that the term *semigroup* is an abuse of notation for integrated semigroups. Indeed, if $(S(t))_{t\geq 0}$ is a once-integrated semigroup, then of course $S(0) \neq \operatorname{id}$ (Remark 5.13), and it only satisfies the functional equation

$$S(r)S(t) = \int_0^r \big(S(t+\tau) - S(\tau)\big) \, d\tau = S(t)S(r) \qquad t, r \geq 0. \tag{5.1}$$

Actually, if $S : [0, \infty) \to \mathcal{L}(X)$ is a strongly continuous function satisfying $S(0) = 0$ and (5.1) such that its *degeneration space*

$$N_S := \{x \in X : S(t)x = 0 \text{ for all } t \geq 0\} = \{0\},$$

then $(S(t))_{t\geq 0}$ is a once-integrated semigroup generated by the operator A defined as: $x \in \operatorname{dom}(A)$ and $Ax = y$ if and only if

$$S(t)x - tx = \int_0^t S(r)y \, dr \qquad t \geq 0;$$

see [115, Theorem 3.6]. In this case, A is densely defined if and only if $(S(t))_{t\geq 0}$ is even a C_0-semigroup [10, Corollary 3.3.11]. Furthermore, if $(S(t))_{t\geq 0}$ is exponentially bounded, then by [115, Proposition 3.10] $(\omega, \infty) \subseteq \rho(A)$ and

$$\mathcal{R}(\lambda, A) = \lambda \int_0^\infty e^{-\lambda t} S(t)\, dt \qquad \lambda > \omega.$$

In this way, we get a one-to-one relationship between the semigroup, the generator, and the resolvent. On the other hand, if the degeneration space $N_S \neq \{0\}$, then there still exists a family of pseudo-resolvents $(R(\lambda))_{\lambda > \omega}$ such that

$$R(\lambda) = \lambda \int_0^\infty e^{-\lambda t} S(t)\, dt \qquad \lambda > \omega.$$

Hence, there exists a unique multi-valued operator A such that $R(\,\cdot\,) = \mathcal{R}(\,\cdot\,, A)$. As one would expect $(S(t))_{t\geq 0}$ is again a once-integrated semigroup generated by A; we refer to [115, Theorem 3.14 and 3.15] for details.

Let us describe how integrated semigroups correspond to solutions of initial value problems [10, Corollary 3.2.11 and Theorem 3.2.13].

Theorem 5.15. *Let A be generator of a k-times integrated semigroup for $k \in \mathbb{N}$ on a Banach space X. Corresponding to $f \in L^1([0, \tau], X)$ with $\tau > 0$, consider the abstract Cauchy problem*

$$\begin{cases} \dot{u}(t) = Au(t) + f(t) & \text{for all } t \geq \tau, \\ u(0) = x. \end{cases} \tag{ACP$_f$}$$

The following assertions hold.

(a) *If $x \in \operatorname{dom}(A^{k+1})$, then (ACP$_f$) has a unique classical solution for $f = 0$.*
(b) *If $f \in C^k([0, \tau], X)$ and the vectors*

$$x_0 = x, x_{j+1} = Ax_j + f^{(j)}(0) \qquad j \in \{0, \dots, k-1\}$$

all lie in $\operatorname{dom}(A)$, then (ACP$_f$) has a unique mild solution.
(c) *If $f \in C^{k+1}([0, \tau], X)$ and the vectors*

$$x_0 = x, x_{j+1} = Ax_j + f^{(j)}(0) \qquad j \in \{0, \dots, k\}$$

all lie in $\operatorname{dom}(A)$, then (ACP$_f$) has a unique classical solution.

Conversely, if $\rho(A) \neq \varnothing$ and (5.1) has a unique classical exponentially bounded solution for $f = 0$ and every $x \in \operatorname{dom}(A^{k+1})$, then A is the generator of an exponentially bounded k-times integrated semigroup on X.

We close this subsection by noting that there is also a variety of perturbation results for integrated semigroups available; we refer the reader to the references [75, 76, 81, 100, 104] for details.

6. Positive semigroups

6.1. **Function spaces and Banach lattices.** Before we discuss the topic of positive semigroups in detail, let us give a brief primer on Banach lattices in this subsection; these spaces provide the theoretical framework in which much of the theory of positive semigroups is developed.

The order on function spaces. A common feature of many classical function spaces such as, for instance, $L^p(\Omega, \mu)$ for a measure space (Ω, μ) and $p \in [1, \infty]$ or the space $\mathrm{C_b}(\Omega)$ of bounded continuous functions over a topological space Ω, is that they admit a partial order that is compatible with the vector space structure. On L^p-spaces this order is defined pointwise almost everywhere and on $\mathrm{C_b}(\Omega)$ it is given pointwise. Moreover, any two functions f, g in such a space have a *supremum* $f \vee g$ and an *infimum* $f \wedge g$. Note that it is not necessary to define this suprema and infima pointwise (or pointwise almost everywhere). Rather, one can define them to be the smallest upper bound and the largest lower bound of f and g with respect to the partial order on the function space and then derive that they coincide with the pointwise (almost everywhere) supremum and infimum.

While this distinction might seem overly nuanced at first glance, the distinction becomes important when one considers suprema and infima of infinite sets. In addition, and more importantly for our purposes, defining the suprema and infima of two vectors with respect to the order on the vector space opens the door for an axiomatic treatment of ordered vector spaces where such finite suprema and infima always exist. These are the so-called *vector lattices* which are sometimes also referred to as *Riesz spaces*. In such a vector lattice E one can define the *modulus* of each vector f as $|f| := f \vee (-f)$. For the theory of Riesz spaces, we refer, for instance, to the books [97, 125, 126]. The theory of more general ordered vector spaces is also classical. Studying them specifically by embedding them into vector lattices has seen a lot of recent activity within the theory of the so-called *pre-Riesz spaces*, see the monograph [77]. Throughout we will use the notation E_+ for the cone of positive elements of a vector lattice E.

Banach lattices and positive operators. If a vector lattice E is, at the same time, a Banach space and the norm satisfies the condition

$$|f| \leq |g| \quad \Rightarrow \quad \|f\| \leq \|g\|$$

for all $f, g \in E$, then E is called a *Banach lattice*. Banach lattices can, as indicated above, seen as axiomatic generalizations of classical function spaces. For a detailed treatment, we refer to the monographs [5, 103, 112, 121]. The interaction of Banach lattices with other subjects in infinite-dimensional analysis is explained in a lot of detail in the monograph [4].

A bounded linear operator T on a Banach lattice E is called *positive*, which we notate by $T \geq 0$, if $TE_+ \subseteq E_+$. A deep structure theory of positive operators on Banach lattices is available and can be found in the aforementioned monographs. Moreover, motivated by the classical Perron–Frobenius theorem on positive matrices, a far-reaching spectral theory of positive operators on Banach lattices has been developed. See for instance [62] for a recent overview and [66] for a more classical survey.

6.2. **Positivity.** We are mainly interested in semigroups that consist of positive operators.

Definition 6.1. A semigroup $(T(t))_{t\geq 0}$ on a Banach lattice E is called *positive* (or *positivity preserving*) if $T(t) \geq 0$ for every $t \in [0, \infty)$.

For C_0-semigroups a very detailed theory of positivity is known. We refer, in particular, to the classical book [106] and the more recent one [21]. It is not difficult to characterize the positivity of C_0-semigroups in terms of the resolvent:

Proposition 6.2. *Let $T = (T(t))_{t\geq 0}$ be a C_0-semigroup with generator A on a Banach lattice E. The following are equivalent:*

(i) *The semigroup T is positive.*

(ii) *For all sufficiently large $\lambda \in \mathbb{R}$ the resolvent $\mathcal{R}(\lambda, A)$ is a positive operator.*

The implication "(i) $\Rightarrow$ (ii)" is a consequence of the representation of the resolvent as the Laplace transform of the semigroup and the converse implication follows from Euler's formula for the semigroup in terms of the resolvent. For semigroups with time regularity other than C_0, a similar equivalence remains true if those formulas are valid in an appropriate topology τ on E and the positive cone E_+ is closed with respect to τ. Characterizing positivity in terms of the generator rather than the resolvent is a more delicate matter. For C_0-semigroups this can be done by an abstract version of Kato's inequality as shown in [7]

A particular focus in the theory of positive semigroups is on their long-term behaviour. For this topic we refer, in addition to the aforementioned books [21,106], to the monographs [47,105].

6.3. **Irreducible and positivity improving semigroups.** Irreducibility of a semigroup means, loosely speaking, that for every positive initial value the orbit "visits" each point in the space at some time. To make this notion precise, we first need the concept of an *ideal* in a Banach lattice.

Ideals and quasi-interior points. In classical function spaces, it is often useful to consider a subspace of functions that vanish on a certain set. In the more general setting of Banach lattices, this can be made precise by using the notion of *ideals*:

Definition 6.3. Let E by a vector lattice. An *ideal* (or, more precisely, an *order ideal*) in E is a vector subspace $I \subseteq E$ with the following property: if $x \in E$ and $y \in I$ satisfy $|x| \leq |y|$, then also $x \in I$.

A simple reformulation of the definition of ideals is as follows: a vector subspace $I \subseteq E$ is an ideal if and only if $x \in I$ whenever $y \in I$ and $x \in E$ are such that $0 \leq x \leq y$. The following examples indicate how one should think intuitively about closed ideals in a Banach lattice:

Examples 6.4.

(a) If (Ω, μ) is a σ-finite measure space and $p \in [1, \infty)$, then the closed ideals in $L^p(\Omega, \mu)$ are precisely the sets of the form
$$\big\{f \in L^p(\Omega, \mu) \,|\, f(\omega) = 0 \text{ for almost all } \omega \in A\big\},$$
where $A \subseteq \Omega$ is measurable [21, Proposition 10.15].

(b) Let L be a locally compact Hausdorff space. Then the closed ideals in $C_0(L)$ are precisely the sets of the form
$$\big\{f \in C_0(L) \,|\, f(\omega) = 0 \text{ for all } \omega \in A\big\},$$
where $A \subseteq L$ is closed [21, Proposition 10.14].

(c) The space c_0 of sequences that converge to 0 is a closed ideal in the Banach lattice c of convergent sequences and also in the Banach lattice ℓ^∞ of bounded sequences.

These examples illustrate the concept of closed ideals – however, there are many more non-closed ideals:

Examples 6.5.

(a) If (Ω, μ) is a finite measure space and let $1 \leq p < q \leq \infty$. Then $L^q(\Omega, \mu)$ is an ideal in $L^p(\Omega, \mu)$ that is not closed unless $L^p(\Omega, \mu)$ is finite-dimensional.

(b) If $1 \leq p < q \leq \infty$, then the sequence space ℓ^p is a non-closed ideal in ℓ^q.

(c) The space $C([0, 1])$ is not an ideal in $L^p([0, 1])$ for any $p \in [1, \infty]$.

(d) The space c_{00} of sequences with only finitely many non-zero entries is a non-closed ideal in each of the Banach lattices c_0, c, and ℓ^p for $p \in [1, \infty]$.

(e) Let E be a Banach lattice and $u \in E_+$. Then the set

$$E_u := \{x \in E \mid |x| \leq cu \text{ for some } c \in [0, \infty)\}$$

is the smallest ideal in E that contains u. It is called the *principal ideal* generated by u. This ideal is only closed in general.

If one takes $u = \mathbb{1}$ in the space $E = L^p(\Omega, \mu)$ for some $p \in [1, \infty]$ and a finite measure space (Ω, μ), then $E_u = L^\infty(\Omega, \mu)$ – which is a case that has already occurred in Example (a).

Irreducible semigroups. By using closed ideals, one can define the concept *irreducibility* in a quite general setting:

Definition 6.6. A semigroup $T = (T(t))_{t \geq 0}$ on a Banach lattice E is called *irreducible* if there are no closed ideals $I \subseteq E$ that are invariant under T except for the trivial ones $\{0\}$ and E.

Similarly, a positive linear operator $S \in \mathcal{L}(E)$ is called *irreducible* if there are no closed ideals $I \subseteq E$ invariant under T except for the trivial ones $\{0\}$ and E.

Note that this is different from a stronger irreducibility notion that occurs, for instance, in representation theory. There one considers invariant closed vector subspaces rather than only invariant closed ideals.

For positive semigroups, irreducibility can be characterized by testing against positive vectors and positive functionals. To state the theorem let us recall the following notion: a Banach lattice E is said to have *order continuous norm* if every decreasing net in E_+ with infimum 0 converges in norm to 0. Typical examples for Banach lattices with order continuous norm are all L^p-spaces for $p \in [1, \infty)$ and the space c_0 of sequences that converge to 0.

Theorem 6.7. *Let $T = (T(t))_{t \geq 0}$ be a positive semigroup on a Banach lattice E. Consider the following assertions:*

(i) *The semigroup T is irreducible.*
(ii) *For all non-zero $f \in E_+$ and all non-zero $\varphi \in E'_+$ there exists $t \in [0, \infty)$ such that $\langle \varphi, T(t)f \rangle \neq 0$.*
(iii) *For all non-zero $0 \leq f, g \in E$ there exists a time $t \in [0, \infty)$ such that $f \wedge T(t)g \neq 0$.*

One always has (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). *If E has an order continuous norm, all assertions are equivalent.*

Proof. "(i) $\Rightarrow$ (ii)": Assume that (ii) does not hold. Then we can find a non-zero vector $f \in E_+$ and a non-zero functional $\varphi \in E'_+$ such that $\langle \varphi, T(t)f \rangle = 0$ for all $t \in [0, \infty)$. The set

$$I := \{g \in E \mid \langle \varphi, T(t)\,|g| \rangle = 0 \text{ for all } t \in [0, \infty)\}.$$

is a T-invariant closed ideal in E and $I \neq \{0\}$ as $f \in I$. But we also have $I \neq E$, since otherwise

$$0 \leq |\langle \varphi, g \rangle| \leq \langle \varphi, |g| \rangle = \langle \varphi, T(0)\,|g| \rangle = 0$$

for all $g \in E$, which is a contradiction to $\varphi \neq 0$. Hence, T is not irreducible.

"(ii) $\Rightarrow$ (i)": Assume that T is not irreducible and let $\{0\} \subsetneq I \subsetneq E$ be a T-invariant closed ideal in E. A quotient space argument shows that there exists a non-zero functional $0 \leq \varphi \in E'$ which vanishes on I. Moreover, we can find a non-zero vector $0 \leq f \in I$ since $I \neq \{0\}$. Since the semigroup leaves I invariant ones has $T(t)f \in I$ and in turn, $\langle \varphi, T(t)f \rangle = 0$ for all $t \in [0, \infty)$.

"(i) $\Rightarrow$ (iii)": Assume that (iii) is not true. Then we can find non-zero vectors $0 \le f, g \in E$ such that $f \wedge T(t)g = 0$ for all $t \in [0, \infty)$. We define

$$I := \{h \in E : \; f \wedge T(t)\,|h| = 0 \text{ for all } t \in [0, \infty)\}.$$

Then I is a closed ideal in E and $I \ne \{0\}$ since $g \in I$. On the other hand, we have $0 \ne f \wedge f = f \wedge T(0)\,|f|$, so $f \notin I$; this proves that $I \ne E$. Finally, we note that I is T-invariant: for every $h \in I$ and $s \in [0, \infty)$ we have

$$0 \le f \wedge T(t)\,|T(s)h| \le f \wedge T(t+s)\,|h| = 0$$

for all $t \in [0, \infty)$, so $T(s)h \in I$. Hence, T is not irreducible.

"(iii) $\Rightarrow$ (i)": Now assume that E has order continuous norm and suppose that T is not irreducible. Then there exists a T-invariant closed ideal $\{0\} \subsetneq I \subsetneq E$ in E. As I has order continuous norm, I is automatically a so-called *band* in E, see [103, Corollary 2.4.4]. Hence, we can find a non-zero vector $0 \le g \in I$ and a non-zero vector $0 \le f \in E$ which is disjoint to all elements of I. Due to the invariance of I, this implies $f \wedge T(t)g = 0$ for all $t \in [0, \infty)$. ☐

The equivalence of (i) and (ii) in Theorem 6.7 is well-known for C_0-semigroups, see for instance [106, Definition C-III-3.1 on p. 306], but as the argument above shows it remains true without any time regularity assumption. We note in passing that in [106, Definition C-III-3.1 on p. 306] it is also claimed that (iii) is equivalent to (i) and (ii) for C_0-semigroups on arbitrary Banach lattices. This is not correct, though, as the following example shows:

Remark 6.8. Let $E = C([0,1])$, let $\lambda \in E'$ denote the Lebesgue measure and fix an arbitrary function $u \in E$ which has integral $\langle \lambda, u \rangle = 1$ and which satisfies $u(0) = u(1) = 0$ and $u(\omega) > 0$ for all $\omega \in (0,1)$. The operator $P := u \otimes \lambda \in \mathcal{L}(E)$ is a projection and we consider the C_0-semigroup $T = (T(t))_{t \ge 0} := (e^{tP})_{t \in [0,\infty)}$ generated by P. Note that

$$T(t) = e^{tP} = \operatorname{id} + (e^t - 1)P \qquad \text{for all } t \in [0, \infty)$$

since P is a projection. For all non-zero $0 \le f, g \in E$ we choose, for instance, $t = 1$ and obtain $T(t)g \ge (e-1)Pg \ge Pg = \langle \lambda, g \rangle u$ and thus,

$$f \wedge T(t)g \ge f \wedge \big(\langle \lambda, g \rangle u\big) > 0.$$

Yet, T is not irreducible since the ideal $I := \{f \in E \,|\, f(0) = f(1) = 0\}$ is T-invariant.

For a positive C_0-semigroup T on a Banach lattice E, irreducibility can be characterized in terms of the resolvent: if A denotes the generator of T and $\lambda > s(A)$, then T is irreducible if and only if the operator $\mathcal{R}(\lambda, A)$ is irreducible if and only if $\mathcal{R}(\lambda, A)$ maps every non-zero vector $f \in E_+$ to a quasi-interior point; see for instance [106, Definition C-III-3.1(i), (iv) and (v) on p. 306] or [21, Proposition 14.10 on pp. 222–223]. From this one can easily derive that the irreducibility of a positive C_0-semigroups implies that none of the semigroup operators has a positive non-zero vector in its kernel; see for instance [106, Theorem C-III-3.2(a) on p. 306]. Yet, we show now that the latter result remains true without any time regularity. This result is, to the best of our knowledge, new and the proof is a bit more involved. Note that we have to exclude that case $\dim E = 1$ since in the one-dimensional case the semigroup which is 0 for all $t > 0$ is irreducible; in the C_0-case this subtlety cannot occur.

Theorem 6.9. *Let $T = (T(t))_{t \ge 0}$ be a positive semigroup on a Banach lattice E. If $\dim E \ge 2$, then one has $T(t)f \ne 0$ for every non-zero $f \in E_+$ and every $t \in [0, \infty)$.*

Proof. We proceed in three steps:

Step 1: We show that there exists a time $t > 0$ such that $T(s) \neq 0$ for all $s \in (0, t]$. Assume the contrary. Then we can find a sequence of times $s_n \downarrow 0$ such that $T(s_n) = 0$ for all n. By the semigroup law, this implies that $T(s) = 0$ for all $s \in (0, \infty)$. Since $\dim E \geq 2$ there exists a closed ideal $\{0\} \subsetneq I \subsetneq E$ and it follows from $T(s) = 0$ for all $s > 0$ that I is T-invariant, which contradicts the irreducibility of T.

Step 2: We show that if $f \in E_+$ is non-zero and $T(t)f = 0$ for some $t \in [0, \infty)$, then $T(s) = 0$ for all $s \geq t$. So assume that such f and t are given and observe that $T(s)f = 0$ for all $s \geq t$ by the semigroup law. Now fix $s \in [t, \infty)$ and let $\psi \in E'_+$. If $\varphi := T(s)'\psi \in E'_+$ were non-zero we could find, by Theorem 6.7, a time $r \geq 0$ such that the number $\langle \psi, T(s + r)f \rangle = \langle \varphi, T(r)f \rangle$ is non-zero – but this cannot be true since $s + r \geq t$. So $T(s)'\psi = 0$ for all $\psi' \in E'_+$ and hence $T(s) = 0$, as claimed.

Step 3: We show the conclusion of the theorem. It follows from Steps 1 and 2 that there exists a time $t > 0$ such that $T(s)f \neq 0$ for all $s \in (0, t]$ and all non-zero $f \in E_+$. It thus follows from the semigroup law and the positivity of the semigroup that $T(s)f \neq 0$ for all non-zero $f \in E_+$, all $s \in (0, nt]$ and every integer $n \geq 1$. This proves the theorem. ☐

Positivity improving semigroups. Many concrete semigroups have the property that, for any positive non-zero initial vector, they instantly spread the "mass" of the vector throughout the space. This behaviour can be described mathematically in the following way:

Definition 6.10. A semigroup $T = (T(t))_{t \geq 0}$ on a Banach lattice E is called *positivity improving* or *strongly positive* if for every $t \in (0, \infty)$ and every non-zero $f \in E_+$, the vector $T(t)f$ is a quasi-interior point.

It is an intriguing observation that, for positive semigroups, irreducibility together with analyticity assumptions already implies that the semigroup is positivity improving.

Theorem 6.11. *Let $T = (T(t))_{t \geq 0}$ be a positive and irreducible semigroup on a complex Banach lattice E and let $\dim E \geq 2$. Assume that there exists a number $\theta > 0$ and an analytic mapping $\tilde{T}$ from the open sector*

$$\{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \theta\}$$

to $\mathcal{L}(E)$ that coincides with T on $(0, \infty)$. Then T is positivity improving.

For C_0-semigroups this is a classical result (and also holds if $\dim E = 1$), see for instance [106, Theorem C-III-3.2(b) on p. 306]. For the more general case considered in Theorem 6.11 the argument has to be adapted a bit; the details can be found in the recent paper [59, Theorem A.1]. More precisely, it is shown there that, for a positive semigroup which extends to an analytic mapping on a sector, and for $f \in E_+$ and $\varphi \in E'_+$ one either has $\langle \varphi, T(t)f \rangle > 0$ for all $t \in (0, \infty)$ or $\langle \varphi, T(t)f \rangle = 0$ for all $t \in (0, \infty)$. Together with Theorem 6.9 this readily gives Theorem 6.11.

Persistently irreducible semigroups. Let T be a semigroup on a Banach lattice E. Irreducibility of T means, according to Definition 6.6, that no non-trivial closed ideal is invariant under T. On some occasions it can be useful to know an even stronger property, namely that no non-trivial closed ideal is invariant under any tail of T:

Definition 6.12. Let $T = (T(t))_{t \geq 0}$ be a semigroup on a Banach lattice E.

(a) A set $S \subseteq E$ is called *eventually invariant* under T if there exists a time $t_0 \geq 0$ such that $T(t)I \subseteq I$ for all $t \geq t_0$.

(b) The semigroup T is called *persistently irreducible* if $\{0\}$ and E are the only closed ideals in E that are eventually invariant under E.

Persistent irreducibility was recently introduced and studied in [15] in the context of eventually positive semigroups. The focus in the reference [15] was on the C_0-case. There it was also proved the persistent irreducibility is equivalent to irreducibility for positive C_0-semigroups [15, Proposition 3.5]. Let us now show that this remains true without any time regularity assumption. Again, one has to exclude the one-dimensional case to avoid the semigroup T that satisfies $T(t) = 0$ for all $t \in (0, \infty)$.

Theorem 6.13. *Let $T = (T(t))_{t \geq 0}$ be a positive semigroup on a Banach lattice E. Consider the following assertions:*

(i') *The semigroup T is irreducible.*
(i) *The semigroup T is persistently irreducible.*
(ii) *For all non-zero $f \in E_+$, all non-zero $\varphi \in E'_+$ and all $t_0 \geq 0$ there exists $t \in [t_0, \infty)$ such that $\langle \varphi, T(t)f \rangle \neq 0$.*
(iii) *For all non-zero $0 \leq f, g \in E$ and all $t_0 \geq 0$ there exists a time $t \in [t_0, \infty)$ such that $f \wedge T(t)g \neq 0$.*

One always has (i') $\Leftrightarrow$ (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). *If E has order continuous norm, all assertions are equivalent.*

Proof. "(i) $\Rightarrow$ (i')": This implication is clear.

"(i') $\Rightarrow$ (ii)": Let T be irreducible, let $f \in E_+$ and $\varphi \in E'_+$ both be non-zero and let $t_0 \in [0, \infty)$. As $\dim E \geq 2$, the irreduciblity of T implies that $T(t_0)f \neq 0$ according to Theorem 6.9. Hence, by the weak characterization of irreducibility in Theorem 6.7 there exists a time $t \geq 0$ such that

$$\langle \varphi, T(t + t_0)f \rangle = \langle \varphi, T(t)T(t_0)f \rangle > 0.$$

"(ii) $\Rightarrow$ (i)": If (i) fails, then there exists a closed ideal $\{0\} \subsetneq I \subsetneq E$ and $t_0 \geq 0$ such that $T(t)I \subseteq I$ for all $t \geq t_0$. As I is proper and non-zero, there exists non-zero $f \in E_+$ and a non-zero $\varphi \in E'_+$ that vanishes on I. In particular,

$$\langle \varphi \, , \, T(t)f \rangle = 0$$

for all $t \geq t_0$, a contradiction.

"(i') $\Rightarrow$ (iii)": Let T be irreducible, let $f, g \in E_+$ and $\varphi \in E'_+$ both be non-zero and let $t_0 \in [0, \infty)$. Again, the irreducibility of T together with $\dim E \geq 2$ imply, according to Theorem 6.9, that $T(t_0)g \neq 0$. So it follows from Theorem 6.7 that there exists a time $t \geq 0$ such that

$$f \wedge T(t + t_0)g = f \wedge (T(t)T(t_0)g) \neq 0.$$

"(iii) $\Rightarrow$ (i')": Now assume that E has order continuous norm. Then the desired implication follows from Theorem 6.7. ☐

6.4. **Relatively uniformly continuous semigroups.** Since vector-valued integrals and differentials are main tools in the study of various classes of operator semigroups, some kind of topology is typically necessary to give meaning to these integrals. However, within the framework of positive semigroups, an alternative is possible. For semigroups defined on a vector lattice E one can use order theoretic rather than topological notions of convergence to define integrals and differentials. This makes it possible to develop a rich theory of (positive) operator semigroups on vector lattices without any norm or topology.

Let us recall the relevant order-theoretic notions first:

Definition 6.14. Let E be an Archimedean vector lattice. A net (f_j) in E is said to be *relatively uniformly convergent* or, for short, *ru-convergent* to a vector $f \in E$ if there exists a vector $u \in E_+$ such that for every $\varepsilon > 0$, there exists an index j_0 such that $|f_j - f| \leq \varepsilon u$ for all $j \geq j_0$.

In this case, the vector u is called a *regulator* of the ru-convergence of (f_j) to f.

Based on the notion of ru-convergence one can define *relatively uniformly continuous semigroups* or, for short, *ruc-semigroups*: a semigroup $T = (T(t))_{t \geq 0}$ of positive linear maps on an Archimedean vector lattice E is called a *ruc-semigroup* if, for each $f \in E$, the net $(T(t)f)_{t \in (0,\infty)}$ converges relatively uniformly to f, where the index set $(0, \infty)$ of the net is endowed with the converse of the usual order of real numbers.

Relatively uniformly continuous semigroups were introduced in [78] and generation theorems for them were proved in [79]. If the underlying vector lattice E is even a Banach lattice, the notions of ruc-semigroups and C_0-semigroups are both defined and one can compare both notions. Indeed, it is not difficult to see that if a positive semigroup on a Banach lattice E is a ruc-semigroup, then it is also C_0-semigroup. Conversely, it was shown in [64] that a positive C_0-semigroup on E is a ruc-semigroup if and only if every orbit of the semigroup is order bounded on the time interval $[0, 1]$. Since the order boundedness of orbits is a way to describe a so-called *maximal inequality* for the semigroup, this shows a connection between the theory of ruc-semigroups and harmonic analysis.

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